

**BHARATHIDASANAR MATRIC HIGHER  
SECONDARY SCHOOL**

**ARAKKONAM**

**XII – MATHEMATICS**

**MATERIAL – 5 & 6 UNITS (6 MARKS & 10 MARKS)**

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## 5. DIFFERENTIAL CALCULUS

### APPLICATIONS – I

#### Example sums:

1. The length  $l$  metres of a certain metal rod at temperature  $\theta^{\circ} C$  is given by  $l = 1 + 0.00005 \theta + 0.0000004 \theta^2$ .

Determine the rate of change of length in  $\text{mm}/^{\circ} C$  when the temperature is (i)  $100^{\circ} C$  and (ii)  $400^{\circ} C$ .

#### Solution:

The rate of change of length means  $\frac{dl}{d\theta}$ .

Since length  $l = 1 + 0.00005 \theta + 0.0000004 \theta^2$ ,

$$\frac{dl}{d\theta} = 0.00005 + 0.0000008 \theta .$$

(i) when  $\theta = 100^{\circ} C$

$$\begin{aligned} \frac{dl}{d\theta} &= 0.00005 + (0.0000008) (100) \\ &= 0.00013 \text{ m}/^{\circ} C \\ &= 0.13 \text{ mm}/^{\circ} C \end{aligned}$$

(ii) When  $\theta = 400^{\circ} C$

$$\begin{aligned} \frac{dl}{d\theta} &= 0.00005 + (0.0000008) (400) \\ &= 0.00037 \text{ m}/^{\circ} C \end{aligned}$$

$$= 0.37 \text{ mm/}^\circ \text{ C}$$

2. The luminous intensity  $I$  candelas of a lamp at varying voltage  $V$  is given by:

$I = 4 \times 10^{-4} V^2$ . Determine the voltage at which the light is increasing at a rate of 0.6 candelas per volt.

**Solution:** The rate of change of light with respect to voltage is given by

$$\frac{dI}{dV}$$

$$\text{Since } I = 4 \times 10^{-4} V^2$$

$$\frac{dI}{dV} = 8 \times 10^{-4} V.$$

When the light is increasing at 0.6 candelas per volt then

$$\frac{dI}{dV} = + 0.6.$$

Therefore we must have  $+ 0.6 = 8 \times 10^{-4} V$ , from which,

$$\begin{aligned} \text{Voltage } V &= \frac{0.6}{8 \times 10^{-4}} \\ &= 0.075 \times 10^4 \\ &= 750 \text{ Volts.} \end{aligned}$$

3. The distance  $x$  meters described by a car in time  $t$  seconds is given by:  $x = 3t^3 - 2t^2 + 4t - 1$ . Determine the velocity and acceleration when

(i)  $t = 0$  and (ii)  $t = 1.5 \text{ s}$

**Solution:**

$$\text{Distance } x = 3t^3 - 2t^2 + 4t - 1$$

$$\text{Velocity } v = \frac{dx}{dt} = 9t^2 - 4t + 4 \text{ m/s}$$

$$\text{Acceleration } a = \frac{d^2x}{dt^2} = 18t - 4 \text{ m/s}^2$$

(i) When time  $t = 0$

$$\text{Velocity } v = 9(0)^2 - 4(0) + 4 = 4 \text{ m/s and}$$

$$\text{Acceleration } a = 18(0) - 4 = -4 \text{ m/s}^2$$

(ii) When time  $t = 1.5 \text{ sec}$

$$\text{Velocity } v = 9(1.5)^2 - 4(1.5) + 4 = 18.25 \text{ m/sec and}$$

$$\text{Acceleration } a = 18(1.5) - 4 = 23 \text{ m/sec}^2$$

4. Supplies are dropped from an helicopter and distance fallen in time  $t$  seconds is given by  $x = \frac{1}{2}gt^2$  where  $g = 9.8 \text{ m/sec}^2$ . Determine the velocity and acceleration of the supplies after it has fallen for 2 seconds.

**Solution:**

$$\begin{aligned} \text{Distance } x &= \frac{1}{2}gt^2 \\ &= \frac{1}{2}(9.8)t^2 = 4.9t^2 \text{ m} \end{aligned}$$

$$\text{Velocity } v = \frac{dx}{dt} = 9.8t \text{ m/sec}$$

$$\text{Acceleration } a = \frac{d^2x}{dt^2} = 9.8 \text{ m/sec}^2$$

When time  $t = 2$  seconds

$$\text{Velocity } v = (9.8)(2) = 19.6 \text{ m/sec and}$$

$$\text{Acceleration } a = 9.8 \text{ m/sec}^2$$



Which is the acceleration due to gravity.

5. The angular displacement  $\theta$  radians of a fly wheel varies with time  $t$  seconds and follows the equation  $\theta = 9t^2 - 2t^3$ . Determine (i) the angular velocity and acceleration of the fly wheel when time  $t = 1$  second and (ii) the time when the angular acceleration is zero.

**Solution:**

(i) angular displacement  $\theta = 9t^2 - 2t^3$  radians.

angular velocity  $\omega = \frac{d\theta}{dt} = 18t - 6t^2 \text{ rad/s}$

When time  $t = 1$  second,

$$\omega = 18(1) - 6(1)^2 = 12 \text{ rad/s}$$

angular acceleration  $= \frac{d^2\theta}{dt^2} = 18 - 12t \text{ rad/s}^2$

when  $t = 1$ , angular acceleration  $= 6 \text{ rad/s}^2$

(ii) Angular acceleration is zero  $\Rightarrow 18 - 12t = 0$ , from which  $t = 1.5 \text{ s}$

6. A boy, who is standing on a pole of height  $14.7 \text{ m}$  throws a stone vertically upwards. It moves in a vertical line slightly away from the pole and

falls on the ground. Its equation of motion in meters and seconds is

$x = 9.8t - 4.9t^2$  (i) Find the time taken for upward and downward motions.

(ii) Also find the maximum height reached by the stone from the ground.

**Solution:**

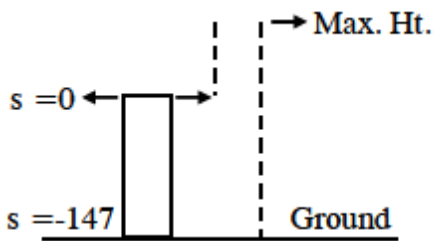
$$(i) x = 9.8 t - 4.9 t^2$$

At the maximum height  $v = 0$

$$v = \frac{dx}{dt} = 9.8 - 9.8 t$$

$$v = 0 \Rightarrow t = 1 \text{ sec}$$

$\therefore$  The time taken for upward motion is 1 sec. For each position  $x$ , there corresponds a time ' $t$ '. The ground position is  $x = -14.7$ , since the top of the pole is taken as  $x = 0$ .



To get the total time, put  $x = -14.7$  in the given equation.

$$\text{i.e., } -14.7 = 9.8 t - 4.9 t^2 \Rightarrow t = -1, 3$$

$\Rightarrow t = -1$  is not admissible and hence  $t = 3$

The time taken for downward motion is  $3 - 1 = 2 \text{ sec 's}$

$$(ii) \text{ When } t = 1, \text{ the position } x = 9.8(1) - 4.9(1) = 4.9 \text{ m}$$

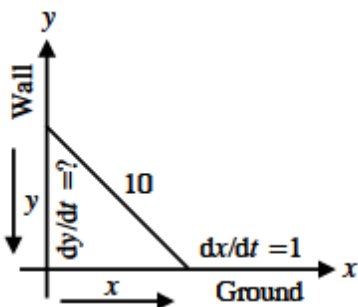
The maximum height reached by the stone = pole height +  $4.9 = 19.6 \text{ m}$

7. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of  $1 \text{ m/sec}$  how fast is the top

of the ladder sliding down the wall when the bottom of the ladder is 6 m from the wall ?

**Solution:**

We first draw a diagram and label it as in *Fig.*



Let  $x$  meters be the distance from the bottom of the ladder to the wall and  $y$  meters be the vertical distance from the top of the ladder to the ground.

Note that  $x$  and  $y$  are both functions of time ' $t$ '.

We are given that  $\frac{dx}{dt} = 1$  m/sec and we are asked

To find  $\frac{dy}{dt}$

When  $x = 6$  m. *Fig.*

In this question, the relationship between  $x$  and  $y$  is given by the

Pythagoras theorem:  $x^2 + y^2 = 100$

Differentiating each side with respect to  $t$ , using chain rule,

we have  $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$  and solving this equation for the derived rate we

obtain,  $\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$

When  $x = 6$ , the Pythagoras theorem gives,  $y = 8$  and so substituting these

values and  $\frac{dx}{dt} = 1$ ,

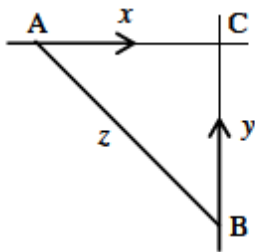
we get  $\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ m/sec}$

The ladder is moving downward at the rate of  $\frac{3}{4} \text{ m/sec}$

8. A car A is travelling from west at 50 km/hr. and car B is travelling towards north at 60 km/hr. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 kilometers and car B is 0.4 kilometers from the intersection?

**Solution:**

We draw *Fig.*



Where C is the intersection of the two roads. At a given time t,

Let  $x$  be the distance from car A to C,

Let  $y$  be the distance from car B to C and

Let  $z$  be the distance between the cars A and B

Where  $x$ ,  $y$  and  $z$  are measured in kilometers.

We are given that  $\frac{dx}{dt} = -50 \text{ km/hr}$  and

$\frac{dy}{dt} = -60 \text{ km/hr}$ .

Note that  $x$  and  $y$  are decreasing and hence the negative sign.

We are asked

To find:  $\frac{dz}{dt}$ .

The equation that relate  $x$ ,  $y$  and  $z$  is given by the Pythagoras theorem  $z^2 = x^2 + y^2$

Differentiating each side with respect to  $t$ ,

$$\text{we have } 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

When  $x = 0.3$  and  $y = 0.4$  km, we get  $z = 0.5$  km and we get

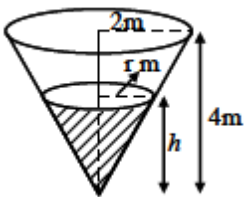
$$\frac{dz}{dt} = \frac{1}{0.5} [0.3(-50) + 0.4(-60)] = -78 \text{ km/hr.}$$

i.e., the cars are approaching each other at a rate of 78 km/hr.

9. A water tank has the shape of an inverted circular cone with base radius 2 meters and height 4 meters. If water is being pumped into the tank at a rate of  $2\text{m}^3/\text{min}$ , find the rate at which the water level is rising when the water is 3m deep.

**Solution:**

We first sketch the cone and label it as in *Fig.*



Let  $V$ ,  $r$  and  $h$  be respectively the volume of the water,

The radius of the cone and the height at time  $t$ ,

Where  $t$  is measured in minutes. *Fig.*

We are given that  $\frac{dV}{dt} = 2m^3/\text{min.}$  and

We are asked to find  $\frac{dh}{dt}$  when  $h$  is 3m.

The quantities  $V$  and  $h$  are related by the equation  $V = \frac{1}{3} \pi r^2 h$ .

But it is very useful to express  $V$  as function of  $h$  alone.

In order to eliminate  $r$  we use similar triangles in *Fig.*

To write  $\frac{r}{h} = \frac{2}{4}$

$$\Rightarrow r = \frac{h}{2} \text{ and the expression for } V \text{ becomes } V = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

Now we can differentiate each side with respect to  $t$  and we have

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

Substituting  $h = 3m$  and  $\frac{dV}{dt} = 2m^3/\text{min.}$

$$\text{we get, } \frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi} \text{ m/min}$$

10. Find the equations of the tangent and normal to the curve  $y = x^3$  at the point (1,1).

**Solution:**

We have  $y = x^3$ ;

Slope  $y' = 3x^2$ .

At the point (1,1),  $x = 1$  and  $m = 3(1)^2 = 3$ .

Therefore equation of the tangent is  $y - y_1 = m(x - x_1)$

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The equation of the normal is  $y - y_1 = -\frac{1}{m}(x - x_1)$

$$y - 1 = \frac{-1}{3}(x - 1) \text{ or } y = \frac{-1}{3}x + \frac{4}{3}$$

11. Find the equations of the tangent and normal to the curve

$$y = x^2 - x - 2 \text{ at the point } (1, -2).$$

**Solution:**

$$\text{We have } y = x^2 - x - 2;$$

$$\text{Slope, } m = \frac{dy}{dx} = 2x - 1.$$

$$\text{At the point } (1, -2), m = 1$$

$$\text{Hence the equation of the tangent is } y - y_1 = m(x - x_1)$$

$$\text{i.e., } y - (-2) = x - 1$$

$$\text{i.e., } y = x - 3$$

$$\text{Equation of the normal is } y - y_1 = \frac{-1}{m}(x - x_1)$$

$$\text{i.e., } y - (-2) = \frac{-1}{1}(x - 1) \text{ or}$$

Differentiating w.r.to  $x$  we get,

$$y + x \frac{dy}{dx} = 0 \text{ or } y = -x - 1$$

12. Find the equation of the tangent at the point  $(a, b)$  to the curve  $xy = c^2$ .

**Solution:**

The equation of the curve is  $xy = c^2$ .

$$\frac{dy}{dx} = \frac{-y}{x} \text{ and } m = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-b}{a}$$

Hence the required equation of the tangent is

$$y - b = \frac{-b}{a}(x - a)$$

$$\text{i.e., } ay - ab = -bx + ab$$

$$bx + ay = 2ab \text{ or}$$

$$\frac{x}{a} + \frac{y}{b} = 2$$

13. Find equations of the tangent and normal at  $\theta = \frac{\pi}{2}$  to the curve

$$x = a(\theta + \sin \theta), \quad y = a(1 + \cos \theta).$$

Solution:

$$\text{We have } \frac{dx}{d\theta} = a(1 + \cos \theta) = 2a \cos^2 \frac{\theta}{2}$$

$$\frac{dy}{d\theta} = -a \sin \theta = -2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$\text{Then } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\tan \frac{\theta}{2}$$

$$\therefore \text{ slope } m = \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = -\tan \frac{\pi}{4} = -1$$

Also for  $\theta = \frac{\pi}{2}$ , the point on the curve is  $\left(a \frac{\pi}{2} + a, a\right)$ .

Hence the equation of the tangent at  $\theta = \frac{\pi}{2}$  is

$$y - a = (-1) \left[ x - a \left( \frac{\pi}{2} + 1 \right) \right]$$

$$\text{i.e. } x + y = \frac{1}{2} a\pi + 2a \text{ or } x + y - \frac{1}{2} a\pi - 2a = 0$$

Equation of the normal at this point is

$$y - a = (1) \left[ x - a \left( \frac{\pi}{2} + 1 \right) \right]$$

$$\text{Or } x - y - \frac{1}{2} a\pi = 0$$

14. Find the equations of tangent and normal to the curve



$16x^2 + 9y^2 = 144$  at  $(x_1, y_1)$  where  $x_1 = 2$  and  $y_1 > 0$ .

**Solution:**

We have  $16x^2 + 9y^2 = 144$

$(x_1, y_1)$  lies on this curve, where  $x_1 = 2$  and  $y_1 > 0$

$$\therefore (16 \times 4) + 9 y_1^2 = 144 \quad \text{or}$$

$$9 y_1^2 = 144 - 64 = 80$$

$$y_1^2 = \frac{80}{9}$$

$$\therefore y_1 = \pm \frac{\sqrt{80}}{3} \text{ .but } y_1 > 0 \therefore y_1 = \frac{\sqrt{80}}{3}$$

$$\therefore \text{the point of tangency is } (x_1, y_1) = \left( 2, \frac{\sqrt{80}}{3} \right)$$

We have  $16x^2 + 9y^2 = 144$

Differentiating w.r.to  $x$  we get  $\frac{dy}{dx} = -\frac{32x}{18y} = -\frac{16}{9} \left( \frac{x}{y} \right)$

$$\therefore \text{the slope at } \left( 2, \frac{\sqrt{80}}{3} \right) = \left( \frac{dy}{dx} \right)_{\left( 2, \frac{\sqrt{80}}{3} \right)}$$

$$= -\frac{16}{9} \times \frac{2}{\frac{\sqrt{80}}{3}} = -\frac{8}{3\sqrt{5}}$$

$$\therefore \text{the equation of the tangent is } y - \frac{\sqrt{80}}{3} = -\frac{8}{3\sqrt{5}} (x - 2)$$

On simplification we get  $8x + 3\sqrt{5}y = 36$

Similarly the equation of the normal can be found as

$$9\sqrt{5}x - 24y + 14\sqrt{5} = 0$$

15. Find the equations of the tangent and normal to the ellipse

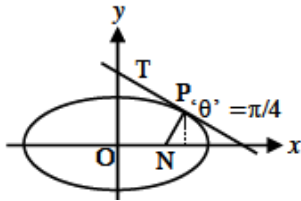
$x = a \cos \theta$  at the point  $\theta = \frac{\pi}{4}$ .

Solution:

$$\text{At } \theta = \frac{\pi}{4}, (x_1, y_1) = \left( a \cos \frac{\pi}{4}, b \sin \frac{\pi}{4} \right) = \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b}{a} \cot \theta$$

$$m = \frac{-b}{a} \cot \frac{\pi}{4} = \frac{-b}{a}$$



Thus the point of tangency is  $\left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$  and the slope is  $m = \frac{-b}{a}$

The equation of the tangent is  $y - \frac{b}{\sqrt{2}} = -\frac{b}{a} \left( x - \frac{a}{\sqrt{2}} \right)$  or

$$bx + ay - ab\sqrt{2} = 0$$

The equation of the normal is  $y - \frac{b}{\sqrt{2}} = \frac{b}{a} \left( x - \frac{a}{\sqrt{2}} \right)$  or

$$(ax - by)\sqrt{2} - (a^2 - b^2) = 0.$$

16. Find equation of the tangent to the parabola,  $y^2 = 20x$

Which forms an angle  $45^\circ$  with the  $x$  - axis.

Solution:

We have  $y^2 = 20x$ . Let  $(x_1, y_1)$  be the tangential point

$$\text{Now } 2yy' = 20 \therefore y' = \frac{10}{y}$$

$$\text{ie., at } (x_1, y_1) \ m = \frac{10}{y_1} \text{ ----- (1)}$$

But the tangent makes an angle  $45^\circ$  with the  $x$  – axis.

$$\therefore \text{Slope of the tangent } m = \tan 45^\circ = 1 \dots (2)$$

From (1) and (2)

$$\frac{10}{y_1} = 1 \Rightarrow y_1 = 10$$

But  $(x_1, y_1)$  lies on  $y^2 = 20x$

$$\Rightarrow y_1^2 = 20x_1$$

$$100 = 20 x_1 \text{ or } x_1 = 5$$

i.e.,  $(x_1, y_1) = (5, 10)$  and

Hence the equation of the tangent at  $(5, 10)$  is

$$y - 10 = 1(x - 5)$$

$$y = x + 5.$$

17. Find the angle between the curves  $y = x^2$  and  $y = (x - 2)^2$  at the point of intersection.

**Solution:**

To get the point of intersection of the curves solve the equation

$$\text{We get } x^2 = (x-2)^2$$

This gives  $x = 1$ . When  $x = 1$ ,  $y = 1$

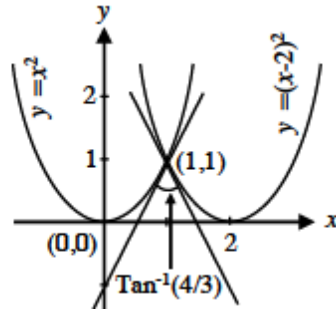
$\therefore$  The point of intersection is  $(1, 1)$

$$\text{Now } y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow m_1 = \left( \frac{dy}{dx} \right)_{(1,1)} = 2$$

$$y = (x - 2)^2 \Rightarrow \frac{dy}{dx} = 2(x - 2)$$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = -2$$



If  $\psi$  is the angle between them, then

$$\tan \psi = \left| \frac{-2-2}{1-4} \right| = \left| \frac{-4}{-3} \right|$$

$$\Rightarrow \psi = \tan^{-1} \frac{4}{3}$$

18. Find the condition for the curves  $ax^2 + by^2 = 1$ ,  $a_1x^2 + b_1y^2 = 1$  to intersect orthogonally.

**Solution:**

If  $(x_1, y_1)$  is the point of intersection,

$$\text{Then } ax_1^2 + by_1^2 = 1; a_1x_1^2 + b_1y_1^2 = 1$$

$$\text{then, } x_1^2 = \frac{b_1 - b}{ab_1 - a_1b}, \quad y_1^2 = \frac{a - a_1}{ab_1 - a_1b} \text{ (by cramer's rule)}$$

For  $ax^2 + by^2 = 1$ ,

$$m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-ax_1}{by_1}$$

For orthogonal intersection, we have  $m_1 m_2 = -1$ .

This gives

$$\left(\frac{-ax_1}{by_1}\right)\left(\frac{-a_1x_1}{b_1y_1}\right) = -1 \text{ or } \frac{aa_1x_1^2}{bb_1y_1^2} = -1$$

$$aa_1x_1^2 + bb_1y_1^2 = 0$$

$$\Rightarrow aa_1\left(\frac{b_1-b}{ab_1-a_1b}\right) + bb_1\left(\frac{a-a_1}{ab_1-a_1b}\right) = 0$$

$$aa_1(b_1 - b) + bb_1(a - a_1) = 0$$

$$\Rightarrow \frac{b_1-b}{bb_1} + \frac{a-a_1}{aa_1} = 0$$

$$\text{Or } \frac{1}{b} - \frac{1}{b_1} + \frac{1}{a_1} - \frac{1}{a} = 0$$

Which is the required condition

19. Show that  $x^2 - y^2 = a^2$  and  $xy = c^2$  cut orthogonally.

**Solution:**

Let  $(x_1, y_1)$  be the point of intersection of the given curves

$$\therefore x_1^2 - y_1^2 = a^2 \text{ and } x_1y_1 = c^2$$

$$x^2 - y^2 = a^2$$

$$\Rightarrow 2x - 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\therefore m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1}{y_1}$$

$$\text{ie., } m_1 = \frac{x_1}{y_1}$$

$$xy = c^2 \Rightarrow y = \frac{c^2}{x} \Rightarrow \frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\therefore m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-c^2}{x_1^2}$$

$$\text{ie., } m_2 = \frac{-c^2}{x_1^2}$$

$$\therefore m_1 m_2 = \left(\frac{x_1}{y_1}\right) \left(\frac{-c^2}{x_1^2}\right) = \frac{-c^2}{x_1 y_1} = \frac{-c^2}{c^2} = -1$$

The curves cut orthogonally

20. Prove that the sum of the intercepts on the co-ordinate axes of any tangent to the curve  $x = a \cos^4 \theta$ ,  $y = a \sin^4 \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$  is equal to  $a$ .

**Solution:**

Take any point ' $\theta$ ' as  $(a \cos^4 \theta, a \sin^4 \theta)$ ,

$$\text{Now } \frac{dx}{d\theta} = -4a \cos^3 \theta \sin \theta ;$$

$$\text{And } \frac{dy}{d\theta} = 4a \sin^3 \theta \cos \theta$$

$$\therefore \frac{dy}{dx} = -\frac{\sin^2 \theta}{\cos^2 \theta}$$

Slope of the tangent at ' $\theta$ ' is

$$(y - a \sin^4 \theta) = -\frac{\sin^2 \theta}{\cos^2 \theta} (x - a \cos^4 \theta)$$

$$\text{Or } x \sin^2 \theta + y \cos^2 \theta = a \sin^2 \theta \cos^2 \theta$$

$$\frac{x}{a \cos^2 \theta} + \frac{y}{a \sin^2 \theta} = 1$$

$$\text{Hence sum of the intercepts} = a \sin^2 \theta + a \cos^2 \theta = a$$

21. Using Rolle's Theorem find the value(s) of  $c$ .

$$(i) f(x) = \sqrt{1 - x^2}, -1 \leq x \leq 1$$

$$(ii) f(x) = (x - a)(b - x), a \leq x \leq b, a \neq b.$$

$$(iii) f(x) = 2x^3 - 5x^2 - 4x + 3, \frac{1}{2} \leq x \leq 3$$

**Solution:**

(i) The function is continuous in  $[-1, 1]$  and differentiable in  $(-1, 1)$ .

$f(1) = f(-1) = 0$  all the three conditions are satisfied.

$$f'(x) = \frac{1}{2} \frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = 0 \Rightarrow x = 0.$$

(Note that for  $x = 0$ , denominator = 1  $\neq 0$ ) Thus the suitable point for which

Rolle's theorem holds is  $c = 0$ .

$$(ii) f(x) = (x - a)(b - x), a \leq x \leq b, a \neq b.$$

$f(x)$  is continuous on  $[a, b]$  and  $f'(x)$  exists at every point of  $(a, b)$ .

$f(a) = f(b) = 0$  All the conditions are satisfied.

$$\therefore f'(x) = (b - x) - (x - a)$$

$$f'(x) = 0 \Rightarrow -2x = -b - a \Rightarrow x = \frac{a+b}{2}$$

The suitable point 'c' of Rolle's Theorem is  $c = \frac{a+b}{2}$

$$(iii) f(x) = 2x^3 - 5x^2 - 4x + 3, \frac{1}{2} \leq x \leq 3$$

$f$  is continuous on  $\left[\frac{1}{2}, 3\right]$  and differentiable in  $\left(\frac{1}{2}, 3\right)$

$f(\frac{1}{2}) = 0 = f(3)$ . All the conditions are satisfied.

$$f'(x) = 6x^2 - 10x - 4$$

$$f'(x) = 0 \Rightarrow 3x^2 - 5x - 2 = 0 \Rightarrow (3x + 1)(x - 2) = 0 \Rightarrow x = -\frac{1}{3} \text{ or } x = 2.$$

$x = -\frac{1}{3}$  does not lie in  $(\frac{1}{2}, 3)$

$\therefore x = 2$  is the suitable 'c' of Rolle's Theorem

22. Verify Rolle's Theorem for the following:

(i)  $f(x) = x^3 - 3x + 3$   $0 \leq x \leq 1$

(ii)  $f(x) = \tan x$ ,  $0 \leq x \leq \pi$

(iii)  $f(x) = |x|$ ,  $-1 \leq x \leq 1$

(iv)  $f(x) = \sin^2 x$ ,  $0 \leq x \leq \pi$

(v)  $f(x) = e^x \sin x$ ,  $0 \leq x \leq \pi$

(vi)  $f(x) = x(x-1)(x-2)$ ,  $0 \leq x \leq 2$

**Solution:**

(i)  $f(x) = x^3 - 3x + 3$   $0 \leq x \leq 1$

$f$  is continuous on  $[0, 1]$  and differentiable in  $(0, 1)$

$$f(0) = 3 \text{ and } f(1) = 1 \therefore f(a) \neq f(b)$$

$\therefore$  Rolle's theorem, does not hold, since  $f(a) = f(b)$  is not satisfied.

Also note that  $f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

There exists no point  $c \in (0, 1)$  satisfying  $f'(c) = 0$ .

(ii)  $f(x) = \tan x$ ,  $0 \leq x \leq \pi$

$f'(x)$  is not continuous in  $[0, \pi]$  as  $\tan x$  tends to  $+\infty$  at  $x = \frac{\pi}{2}$



∴ Rolle's theorem is not applicable.

$$(iii) f(x) = |x|, -1 \leq x \leq 1$$

$f$  is continuous in  $[-1, 1]$  but not differentiable in  $(-1, 1)$  since  $f'(0)$  does not exist.

∴ Rolle's theorem is not applicable.

$$(iv) f(x) = \sin^2 x, 0 \leq x \leq \pi$$

$f$  is continuous in  $[0, \pi]$  and differentiable in  $(0, \pi)$ .  $f(0) = f(\pi) = 0$

(ie.,)  $f$  satisfies hypothesis of Rolle's theorem.

$$f'(x) = 2 \sin x \cos x = \sin 2x$$

$$f'(c) = 0 \Rightarrow \sin 2c = 0 \Rightarrow 2c = 0, \pi, 2\pi, 3\pi, \dots$$

$$\Rightarrow c = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$$

since  $c = \frac{\pi}{2} \in (0, \pi)$ , the suitable  $c$  of Rolle's theorem is  $c = \frac{\pi}{2}$

$$(v) f(x) = e^x \sin x, 0 \leq x \leq \pi$$

$e^x$  and  $\sin x$  are continuous for all  $x$ , therefore the product  $e^x \sin x$  is continuous in  $0 \leq x \leq \pi$ .

$$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x) \text{ exist in } 0 < x < \pi$$

$\Rightarrow f'(x)$  is differentiable in  $(0, \pi)$ .

$$f(0) = e^0 \sin 0 = 0$$

$$f(\pi) = e^\pi \sin \pi = 0$$

∴  $f$  satisfies hypothesis of Rolle's theorem

Thus there exists  $c \in (0, \pi)$  satisfying  $f'(c) = 0 \Rightarrow e^c (\sin c + \cos c) = 0$

$$\Rightarrow e^c = 0 \text{ or } \sin c + \cos c = 0$$

$e^c = 0 \Rightarrow c = -\infty$  which is not meaningful here.

$$\Rightarrow \sin c = -\cos c \Rightarrow \frac{\sin c}{\cos c} = -1 \Rightarrow \tan c = -1 = \tan \frac{3\pi}{2}$$

$\Rightarrow c = \frac{3\pi}{2}$  is the required point.

(vi)  $f(x) = x(x-1)(x-2)$ ,  $0 \leq x \leq 2$ ,

$f$  is continuous in  $[0,2]$  and differentiable in  $(0,2)$

$f(0) = 0 = f(2)$ , satisfying hypothesis of Rolle's theorem

Now  $f'(x) = (x-1)(x-2) + x(x-2) + x(x-1) = 0$

$$\Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{\sqrt{3}}$$

The required  $c$  in Rolle's theorem is  $1 \pm \frac{1}{\sqrt{3}} \in (0,2)$

23. Apply Rolle's theorem to find points on curve

$$y = -1 + \cos x,$$

where the tangent is parallel to  $x$ -axis in  $[0, 2\pi]$ .

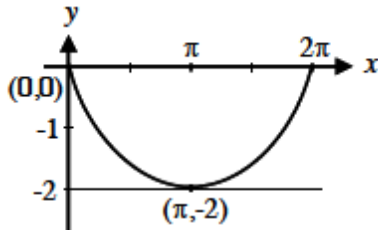
**Solution:**

$f(x)$  is continuous in  $[0, 2\pi]$  and

differentiable in  $(0, 2\pi)$

$f(0) = 0 = f(2\pi)$  satisfying hypothesis

of Rolle's theorem.



Now  $f'(x) = -\sin x = 0 \Rightarrow \sin x = 0$

$x = 0, \pi, 2\pi, \dots$

$x = \pi$ , is the required  $c$  in  $(0, 2\pi)$ . At  $x = \pi$ ,  $y = -1 + \cos \pi = -2$ .

$\Rightarrow$  the point  $(\pi, -2)$  is such that at this point the tangent to the curve is parallel to  $x$ -axis.

24. Verify Lagrange's law of the mean for  $f(x) = x^3$  on  $[-2, 2]$

**Solution:**

$f$  is a polynomial, hence continuous and differentiable on  $[-2, 2]$ .

$f(2) = 2^3 = 8$ ;  $f(-2) = (-2)^3 = -8$

$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$

By law of the mean there exists an element  $c \in (-2, 2)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 = \frac{8 - (-8)}{4} = 4$$

i.e.,  $c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}}$

The required 'c' in the law of mean are  $\frac{2}{\sqrt{3}}$  and  $-\frac{2}{\sqrt{3}}$  as

both lie in  $[-2, 2]$ .

25. A cylindrical hole 4 mm in diameter and 12 mm deep in a metal block is rebored to increase the diameter to 4.12 mm. Estimate the amount of metalre moved.

**Solution:**

The volume of cylindrical hole of radius  $x$  mm and depth 12 mm is given by

$$V = f(x) = 12 \pi x^2$$

$$\Rightarrow f'(c) = 24 \pi c.$$

To estimate  $f(2.06) - f(2)$  :

By law of mean,

$$f(2.06) - f(2) = 0.06 f'(c)$$

$$= 0.06 (24 \pi c), 2 < c < 2.06$$

Take  $c = 2.01$

$$f(2.06) - f(2) = 0.06 \times 24 \pi \times 2.01$$

$$= 2.89 \pi \text{ cubic mm.}$$

26. Suppose that  $f(0) = -3$  and  $f'(x) \leq 5$  for all values of  $x$ , how large can  $f(2)$  possibly be?

**Solution:**

Since by hypothesis  $f$  is differentiable,  $f$  is continuous everywhere.

We can apply Lagrange's Law of the mean on the interval  $[0,2]$ .

There exist atleast one ' $c$ '  $\in (0, 2)$  such that

$$f(2) - f(0) = f'(c) (2 - 0)$$

$$f(2) = f(0) + 2f'(c)$$
$$= -3 + 2f'(c)$$

Given that  $f'(x) \leq 5$  for all  $x$ . In particular we know that  $f'(c) \leq 5$ .

Multiplying both sides of the inequality by 2, we have

$$2f'(c) \leq 10$$

$$f(2) = -3 + 2f'(c) \leq -3 + 10 = 7$$

i.e., the largest possible value of  $f(2)$  is 7.

27. It took 14 sec for a thermometer to rise from  $-19^\circ C$  to  $100^\circ C$  when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly  $8.5^\circ C/\text{sec}$ .

**Solution:**

Let  $T$  be the temperature reading shown in the thermometer at anytime  $t$ . Then  $T$  is a function of time  $t$ . Since the temperature rise is continuous and since there is a continuous change in the temperature the function is differentiable too.

$\therefore$  By law of the mean there exists a ' $t_0$ ' in  $(0, 14)$

such that

$$\frac{T(t_2) - T(t_1)}{t_2 - t_1} = T'(t_0)$$

Here  $T'(t_0)$  is the rate of rise of temperature at  $C$ .

Here  $t_2 - t_1 = 14$ ,  $T(t_2) = 100$ ;  $T(t_1) = -19$

$$T'(t_0) = \frac{100 + 19}{14} = \frac{119}{14} = 8.5C/\text{sec}$$

28. Obtain the Maclaurin's Series for

1)  $e^x$  2)  $\log_e(1+x)$  3)  $\arctan x$  or  $\tan^{-1}x$

**Solution:**

$$(1) f(x) = e^x; f(0) = e^0 = 1$$

$$f'(x) = e^x; f'(0) = 1$$

$$f''(x) = e^x; f''(0) = 1$$

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$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{holds for all } x$$

$$(2) f(x) = \log_e(1+x); f(0) = \log_e 1 = 0$$

$$f'(x) = \frac{1}{1+x}; f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2}; f''(0) = -1$$

$$f'''(x) = \frac{+1.2}{(1+x)^3}; f'''(0) = 2!$$

$$f''''(x) = \frac{-1.2.3}{4}; f''''(0) = -(3!)$$

$$f(x) = \log_e(1+x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2!x^3}{3!} - \frac{3!x^4}{4!} + \dots$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots - 1 < x \leq 1.$$

$$(3) f(x) = \tan^{-1}x; f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \dots ; f'(0) = 1 = 1!$$

$$f''(x) = -2x + 4x^3 - 6x^5 \dots ; f''(0) = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 \dots ; f'''(0) = -2 = -(2!)$$

$$f^{iv}(x) = 24x - 120x^3 \dots ; f^{iv}(0) = 0$$

$$f^v(x) = 24 - 360x^2 \dots ; f^v(0) = 24 = 4!$$

$$\tan^{-1}x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{4!}{5!}x^5 + \dots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \text{holds in } |x| \leq 1.$$

**Example 5.29 :** Evaluate :  $\lim_{x \rightarrow 0} \frac{x}{\tan x}$

**Solution :**  $\lim_{x \rightarrow 0} \frac{x}{\tan x}$  is of the type  $\frac{0}{0}$ .

$$\therefore \lim_{x \rightarrow 0} \frac{x}{\tan x} = \lim_{x \rightarrow 0} \frac{1}{\sec^2 x} = \frac{1}{1} = 1$$

**Example 5.30 :** Find  $\lim_{x \rightarrow +\infty} \frac{\sin \frac{1}{x}}{\tan^{-1} \frac{1}{x}}$  if exists

**Solution :** Let  $y = \frac{1}{x}$  As  $x \rightarrow \infty, y \rightarrow 0$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\sin \frac{1}{x}}{\tan^{-1} \frac{1}{x}} &= \lim_{y \rightarrow 0} \frac{\sin y}{\tan^{-1} y} = \frac{0}{0} \\ &= \lim_{y \rightarrow 0} \left[ \frac{\cos y}{1+y^2} \right] = \frac{1}{1} = 1 \end{aligned}$$

**Example 5.31 :**  $\lim_{x \rightarrow \pi/2} \frac{\log(\sin x)}{(\pi - 2x)^2}$

**Solution :** It is of the form  $\frac{0}{0}$

$$\begin{aligned} \lim_{x \rightarrow \pi/2} \frac{\log(\sin x)}{(\pi - 2x)^2} &= \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sin x} \cos x}{2(\pi - 2x) \times (-2)} \\ &= \lim_{x \rightarrow \pi/2} \frac{\cot x}{-4(\pi - 2x)} = \frac{0}{0} \\ &= \lim_{x \rightarrow \pi/2} \frac{-\operatorname{cosec}^2 x}{-4 \times -2} = \frac{-1}{8} \end{aligned}$$

Note that here l'Hôpital's rule, applied twice yields the result.



**Example 5.32 :** Evaluate :  $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$

**Solution :**  $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$  is the type  $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

**Example 5.33 :** Evaluate :  $\lim_{x \rightarrow 0} \left( \operatorname{cosec} x - \frac{1}{x} \right)$

**Solution :**  $\lim_{x \rightarrow 0} \left( \operatorname{cosec} x - \frac{1}{x} \right)$  is of the type  $\infty - \infty$ .

$$\lim_{x \rightarrow 0} \left( \operatorname{cosec} x - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} \left( = \frac{0}{0} \text{ type} \right) &= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + \cos x - x \sin x} \\ &= \frac{0}{2} = 0 \end{aligned}$$

**Example 5.34 :** Evaluate :  $\lim_{x \rightarrow 0} (\cot x)^{\sin x}$

**Solution :**  $\lim_{x \rightarrow 0} (\cot x)^{\sin x}$  is of the type  $\infty^0$ .

$$\text{Let } y = (\cot x)^{\sin x} \Rightarrow \log y = \sin x \log (\cot x)$$

$$\lim_{x \rightarrow 0} (\log y) = \lim_{x \rightarrow 0} \sin x \log (\cot x)$$

$$= \lim_{x \rightarrow 0} \frac{\log (\cot x)}{\operatorname{cosec} x} \text{ is of the type } \frac{\infty}{\infty}$$

Applying l'Hôpital's rule,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(\cot x)}{\operatorname{cosec} x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\cot x} (-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \frac{0}{1} = 0\end{aligned}$$

$$\text{i.e., } \lim_{x \rightarrow 0} \log y = 0$$

By Composite Function Theorem, we have

$$0 = \lim_{x \rightarrow 0} \log y = \log \left( \lim_{x \rightarrow 0} y \right) \Rightarrow \lim_{x \rightarrow 0} y = e^0 = 1$$

**Caution :** When the existence of  $\lim_{x \rightarrow a} f(x)$  is not known,  $\log \left\{ \lim_{x \rightarrow a} f(x) \right\}$  is meaningless.

**Example 5.35 :** Evaluate  $\lim_{x \rightarrow 0^+} x^{\sin x}$

**Solution :**  $\lim_{x \rightarrow 0^+} x^{\sin x}$  is of the form  $0^0$ .

$$\text{Let } y = x^{\sin x} \Rightarrow \log y = \sin x \log x.$$

Note that  $x$  approaches 0 from the right so that  $\log x$  is meaningful

$$\text{i.e., } \log y = \frac{\log x}{\operatorname{cosec} x}$$

$$\lim_{x \rightarrow 0^+} \log y = \lim_{x \rightarrow 0^+} \frac{\log x}{\operatorname{cosec} x} \text{ which is of the type } \frac{-\infty}{\infty}.$$

Applying l'Hôpital's rule,

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\log x}{\operatorname{cosec} x} &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x \cos x} \left( \text{of the type } \frac{0}{0} \right)\end{aligned}$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \sin x \cos x}{x \sin x - \cos x} = 0$$

$$\text{i.e., } \lim_{x \rightarrow 0^+} \log y = 0$$

By Composite Function Theorem, we have

$$0 = \lim_{x \rightarrow 0^+} \log y = \log \lim_{x \rightarrow 0^+} y \Rightarrow \lim_{x \rightarrow 0^+} y = e^0 = 1$$

**Example 5.36 :**

The current at time  $t$  in a coil with resistance  $R$ , inductance  $L$  and subjected to a constant electromotive force  $E$  is given by  $i = \frac{E}{R} \left( 1 - e^{-\frac{Rt}{L}} \right)$ . Obtain a suitable formula to be used when  $R$  is very small.

**Solution :**

$$\lim_{R \rightarrow 0} i = \lim_{R \rightarrow 0} \frac{E \left( 1 - e^{-\frac{Rt}{L}} \right)}{R} \quad (\text{is of the type } \frac{0}{0})$$

$$= \lim_{R \rightarrow 0} \frac{E \times \frac{t}{L} e^{-\frac{Rt}{L}}}{1} = \frac{Et}{L} \Rightarrow \lim_{R \rightarrow 0} i = \frac{Et}{L} \text{ is the suitable formula.}$$

**Example 5.37 :** Prove that the function  $f(x) = \sin x + \cos 2x$  is not monotonic on the interval  $\left[ 0, \frac{\pi}{4} \right]$ .

**Solution :**

$$\text{Let } f(x) = \sin x + \cos 2x$$

$$\text{Then } f'(x) = \cos x - 2 \sin 2x$$

$$\text{Now } f'(0) = \cos 0 - 2 \sin 0 = 1 - 0 = 1 > 0$$

$$\text{and } f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - 2 \sin 2\left(\frac{\pi}{4}\right)$$

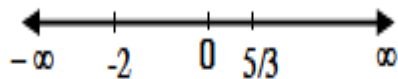
$$= \frac{1}{\sqrt{2}} - 2 \times 1 < 0$$

Thus  $f'$  is of different signs at 0 and  $\frac{\pi}{4}$  Therefore  $f$  is not monotonic on  $\left[0, \frac{\pi}{4}\right]$

**Example 5.38 :** Find the intervals in which  $f(x) = 2x^3 + x^2 - 20x$  is increasing and decreasing.

**Solution :**  $f'(x) = 6x^2 + 2x - 20 = 2(3x^2 + x - 10) = 2(x + 2)(3x - 5)$

Now  $f'(x) = 0 \Rightarrow x = -2$ , and  $x = 5/3$ . The values  $-2$  and  $5/3$  divide the real line (the domain of  $f(x)$ ) into intervals  $(-\infty, -2)$ ,  $(-2, 5/3)$  and  $(5/3, \infty)$ .



**Example 5.39 :** Prove that the function  $f(x) = x^2 - x + 1$  is neither increasing nor decreasing in  $[0,1]$

**Solution :**  $f(x) = x^2 - x + 1$

$$f'(x) = 2x - 1$$

$$f'(x) \geq 0 \text{ for } x \geq \frac{1}{2} \text{ i.e., } x \in \left[\frac{1}{2}, 1\right] \therefore f(x) \text{ is increasing on } \left[\frac{1}{2}, 1\right]$$

$$\text{Also } f'(x) \leq 0 \text{ for } x \leq \frac{1}{2} \Rightarrow x \in \left[0, \frac{1}{2}\right]. \text{ Also } f'(x) \text{ is decreasing on } \left[0, \frac{1}{2}\right]$$

Therefore in the entire interval  $[0,1]$  the function  $f(x)$  is neither increasing nor decreasing.

**Example 5.40:** Discuss monotonicity of the function

$$f(x) = \sin x, x \in [0, 2\pi]$$

The points where the tangent to the graph of the function are parallel to the  $x$  - axis are given by  $f'(x) = 0$ , i.e., when  $x = 2, 3$  Now  $f(2) = 29$  and  $f(3) = 28$ .

Therefore the required points are  $(2, 29)$  and  $(3, 28)$

**Example 5.43 :**

Show that  $f(x) = \tan^{-1}(\sin x + \cos x)$ ,  $x > 0$  is a strictly increasing function in the interval  $\left(0, \frac{\pi}{4}\right)$ .

**Solution :**  $f(x) = \tan^{-1}(\sin x + \cos x)$ .

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} (\cos x - \sin x) = \frac{\cos x - \sin x}{2 + \sin 2x} > 0$$

since  $\cos x - \sin x > 0$  in the interval  $\left(0, \frac{\pi}{4}\right)$

and  $2 + \sin 2x > 0$

$\therefore f(x)$  is strictly increasing function of  $x$  in the interval  $\left(0, \frac{\pi}{4}\right)$

**Example 5.44:**

Prove that

$$e^x > 1 + x \text{ for all } x > 0.$$

**Solution:** Let  $f(x) = e^x - x - 1 \Rightarrow f'(x) = e^x - 1 > 0$  for  $x > 0$

i.e.,  $f$  is strictly increasing function.  $\therefore$  for  $x > 0$ ,  $f(x) > f(0)$

$$\text{i.e., } (e^x - x - 1) > (e^0 - 0 - 1); e^x > x + 1$$

**Example 5.4:**

Prove that the inequality  $(1 + x)^n > 1 + nx$  is true whenever  $x > 0$  and  $n > 1$ .

**Solution:** Consider the difference  $f(x) = (1 + x)^n - (1 + nx)$

$$\text{Then } f'(x) = n(1 + x)^{n-1} - n = n[(1 + x)^{n-1} - 1]$$

Since  $x > 0$  and  $n - 1 > 0$ , we have  $(1 + x)^{n-1} > 1$ , so  $f'(x) > 0$ .

Therefore  $f$  is strictly increasing on  $[0, \infty)$ .

For  $x > 0 \Rightarrow f(x) > f(0)$  i.e.,  $(1 + x)^n - (1 + nx) > (1 + 0)^n - (1 + 0)$

i.e.,  $(1 + x)^n - (1 + nx) > 0$  i.e.,  $(1 + x)^n > (1 + nx)$

**Example 5.46:** Prove that  $\sin x < x < \tan x$ ,  $x \in \left(0, \frac{\pi}{2}\right)$

**Solution:**

Let  $f(x) = x - \sin x$

$$f'(x) = 1 - \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$\therefore f$  is strictly increasing.

For  $x > 0$ ,  $f(x) > f(0)$

$$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x \cdots (1)$$

Let  $g(x) = \tan x - x$

$$g'(x) = \sec^2 x - 1 = \tan^2 x > 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

$\therefore g$  is strictly increasing

$$\text{For } x > 0, f(x) > f(0) \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x \cdots (2)$$

From (1) and (2)  $\sin x < x < \tan x$

**Example 5.47:** Find the critical numbers of  $x^{3/5} (4 - x)$

**Solution:**

$$f(x) = 4x^{3/5} - x^{8/5}$$

$$f'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5}$$

$$= \frac{4}{5} x^{-2/5} (3 - 2x)$$

Therefore  $f'(x) = 0$  if  $3 - 2x = 0$  i.e., if  $x = \frac{3}{2}$ .  $f'(x)$  does not exist when  $x = 0$ .

Thus the critical numbers are 0 and  $\frac{3}{2}$ .

Note that if  $f$  has a local extremum at  $c$ , then  $c$  is a critical number of  $f$ , but

not vice versa.

To find the absolute maximum and absolute minimum values of a continuous function  $f$  on a closed interval  $[a, b]$  :

- (1) Find the values of  $f$  at the critical numbers, of  $f$  in  $(a, b)$ .
- (2) Find the values of  $f(a)$  and  $f(b)$
- (3) The largest of the values from steps 1 and 2 is the absolute maximum value, the smallest of these values is the absolute minimum value.

**Example 5.48:** Find the absolute maximum and minimum values of the function.  $f(x) = x^3 - 3x^2 + 1$ ,  $-1 \leq x \leq 4$

**Solution:** Note that  $f$  is continuous on  $(-1, 4)$

$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since  $f'(x)$  exists for all  $x$ , the only critical numbers of  $f$  are  $x = 0$ ,  $x = 2$ . Both of these critical numbers lie in the interval  $(-1, 4)$  Value of  $f$  at these critical numbers are  $f(0) = 1$  and  $f(2) = -3$ .

The values of  $f$  at the end points of the interval are

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 1 = \frac{1}{8}$$

and  $f(4) = 4^3 - 3 \times 4^2 + 1 = 17$



Comparing these four numbers, we see that the absolute maximum value is  $f(4) = 17$  and the absolute minimum value is  $f(2) = -3$ .

Note that in this example the absolute maximum occurs at an end point, where as the absolute minimum occurs at a critical number.

**Example 5.48(a):** Find the absolute maximum and absolute minimum values of

$$f(x) = x - 2\sin x, 0 \leq x \leq 2\pi.$$

**Solution:**  $f(x) = x - 2\sin x$ , is continuous in  $[0, 2\pi]$

$$f'(x) = 1 - 2\cos x$$

$$f'(x) = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

The value of  $f$  at these critical points are

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3}$$

$$= \frac{5\pi}{3} + \sqrt{3}$$

$$\approx 6.968039$$

The values of  $f$  at the end points are  $f(0) = 0$  and  $f(2\pi) = 2\pi \approx 6.28$

Comparing these four numbers, the absolute minimum is  $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$  and

the absolute maximum is  $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$ . In this example both absolute minimum and absolute maximum occurs at the critical numbers.

Let us now see how the second derivatives of functions help determining the turning nature (of graphs of functions) and in optimization problems. The second derivative test : Suppose  $f$  is continuous on an open interval that contains  $c$ .

(a) If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$ .

(b) If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$ .



**Example 5.49:** Discuss the curve  $y = x^4 - 4x^3$  with respect to local extrema.

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2, f''(x) = 12x^2 - 24x$$

**Example 5.50:** Locate the extreme point on the curve  $y = 3x^2 - 6x$  and determine its nature by examining the sign of the gradient on either side.

**Solution:** Following the above procedure

(i) Since  $y = 3x^2 - 6x$ ,  $\frac{dy}{dx} = 6x - 6$

(ii) At a stationary point,  $\frac{dy}{dx} = 0$ , hence  $x = 1$

(iii) When  $x = 1$ ,  $y = 3(1)^2 - 6(1) = -3$ . Hence the coordinates of the stationary point is  $(1, -3)$ .

If  $x$  is slightly less than 1, say 0.9, then  $\frac{dy}{dx} = 6(0.9) - 6 = -0.6 < 0$ .

If  $x$  is slightly greater than 1, say 1.1 then  $\frac{dy}{dx} = 6(1.1) - 6 = 0.6 > 0$ .

Since the gradient (slope of the curve) changes its sign from negative to positive  $(1, -3)$  is a minimum point.

**Example 5.56:** Show that the volume of the largest right circular cone that can be inscribed in a sphere of radius  $a$  is  $\frac{8}{27}$  (volume of the sphere).

**Solution:** Given that  $a$  is the radius of the sphere and let  $x$  be the base radius of the cone. If  $h$  is the height of the cone, then its volume is

$$\begin{aligned}
 V &= \frac{1}{3} \pi x^2 h \\
 &= \frac{1}{3} \pi x^2 (a + y) \quad \dots(1)
 \end{aligned}$$

where  $OC = y$  so that height  $h = a + y$ .

From the diagram  $x^2 + y^2 = a^2$

Using (2) in (1) we have

$$V = \frac{1}{3} \pi (a^2 - y^2) (a + y)$$

For the volume to be maximum:

$$V' = 0 \Rightarrow \frac{1}{3} \pi [a^2 - 2ay - 3y^2] = 0$$

$$\Rightarrow 3y = +a \text{ or } y = -a$$

$$\Rightarrow y = \frac{a}{3} \text{ and } y = -a \text{ is not possible}$$

$$\text{Now } V'' = -\pi \frac{2}{3} (a + 3y) < 0 \text{ at } y = \frac{a}{3}$$

$\therefore$  the volume is maximum when  $y = \frac{a}{3}$  and the maximum volume is

$$\frac{1}{3} \pi \times \frac{8a^2}{9} (a + \frac{1}{3} a) = \frac{8}{27} (\frac{4}{3} \pi a^3) = \frac{8}{27} (\text{volume of the sphere})$$

**Example 5.57:** A closed (cuboid) box with a square base is to have a volume of 2000 c.c. The material for the top and bottom of the box is to cost Rs. 3 per square cm. and the material for the sides is to cost Rs. 1.50 per square cm. If the cost of the materials is to be the least, find the dimensions of the box.

**Solution:**

Let  $x, y$  respectively denote the length of the side of the square base and depth of the box. Let  $C$  be the cost of the material

$$\text{Area of the bottom} = x^2$$

$$\text{Area of the top} = x^2$$

$$\text{Combined area of the top and bottom} = 2x^2$$

$$\text{Area of the four sides} = 4xy$$

$$\text{Cost of the material for the top and bottom} = 3(2x^2)$$

$$\text{Cost of the material for the sides} = (1.5)(4xy) = 6xy$$

$$\text{Total cost } C = 6x^2 + 6xy \dots(1)$$

$$\text{Volume of the box } V = (\text{area})(\text{depth}) = x^2y = 2000 \dots(2)$$

$$\text{Eliminating } y \text{ from (1) \& (2) we get } C(x) = 6x^2 + \frac{12000}{x} \dots(3)$$

where  $x > 0$ , i.e.,  $x \in (0, +\infty)$  and  $C(x)$  is continuous on  $(0, +\infty)$ .

$$C'(x) = 12x - \frac{12000}{x^2}$$

$$C'(x) = 0 \Rightarrow 12x^3 - 12000 = 0 \Rightarrow 12(x^3 - 10^3) = 0$$

$$\Rightarrow x = 10 \text{ or } x^2 + 10x + 100 = 0$$

$$x^2 + 10x + 100 = 0 \text{ is not possible}$$

$\therefore$  The critical numbers is  $x = 10$ .

$$\text{Now } C''(x) = 12 + \frac{24000}{x^3} \quad ; \quad C''(10) = 12 + \frac{24000}{1000} = 36 > 0$$

$\therefore$   $C$  is minimum at  $(10, C(10)) = (10, 1800)$   $\therefore$  the base length is 10cm and depth is  $y = \frac{2000}{100} = 20$  cm.

### **Example 5.58:**

A man is at a point  $P$  on a bank of a straight river, 3 km wide, and wants to reach point  $Q$ , 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point  $R$  and then run to  $Q$ , or he could row directly to  $Q$ , or he could row to some point  $S$  between  $Q$  and  $R$  and then run to  $Q$ . If he can row at 6 km/h and run at 8 km/h where should he land to reach  $Q$  as soon as possible ?

**Solution :**

Let  $x$  be the distance from  $R$  to  $S$ . Then the running distance is  $8 - x$  and the distance

$$PS = \sqrt{x^2 + 9} . \text{ We know that time} = \frac{\text{distance}}{\text{rate}} .$$

Then the rowing time

$$R_t = \frac{\sqrt{x^2 + 9}}{6} \text{ and the running time } r_t = \frac{(8 - x)}{8}$$

$$\text{Therefore the total time } T = R_t + r_t = \frac{\sqrt{x^2 + 9}}{6} + \frac{(8 - x)}{8}, \quad 0 \leq x \leq 8.$$

Notice that if  $x = 0$ , he rows to  $R$  and if  $x = 8$  he rows directly to  $Q$ .

$$T'(x) = 0 \Rightarrow T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8} = 0 \text{ for critical points.}$$

$$4x = 3\sqrt{x^2 + 9}$$

$$16x^2 = 9(x^2 + 9)$$

$$7x^2 = 81$$

$$\Rightarrow x = \frac{9}{\sqrt{7}} \text{ since } x = -\frac{9}{\sqrt{7}} \text{ is not admissible.}$$

The only critical number is  $x = \frac{9}{\sqrt{7}}$ . We calculate  $T$  at the end point of the domain 0 and 8 and at  $x = \frac{9}{\sqrt{7}}$ .

$$T(0) = 1.5, \quad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33, \text{ and } T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of  $T$  occurs when  $x = \frac{9}{\sqrt{7}}$ , the man should land the boat at a point  $\frac{9}{\sqrt{7}}$  km ( $\approx 3.4$  km) down stream from his starting point.

**Example 5.59:**

Determine the domain of concavity (convexity) of the curve

$$y = 2 - x^2.$$

**Solution:**  $y = 2 - x^2$

$$y' = -2x \text{ and } y'' = -2 < 0 \text{ for } x \in R$$

Here the curve is everywhere concave downwards (convex upwards).

**Example 5.60:**

Determine the domain of convexity of the function  $y = e^x$ .

**Solution :**  $y = e^x$ ;  $y'' = ex > 0$  for  $x$

Hence the curve is everywhere convex downward.

**Example 5.61 :** Test the curve  $y = x^4$  for points of inflection.

**Solution :**  $y = x^4$

$$y'' = 12x^2 = 0 \text{ for } x = 0$$

and  $y'' > 0$  for  $x < 0$  and  $x > 0$

Therefore the curve is concave upward and  $y''$  does not change signs  $y(x)$  passes through  $x = 0$ . Thus the curve does not admit any point of inflection.

**Note :** The curve is concave upward in  $(-\infty, 0)$  and  $(0, \infty)$ .

**Example 5.62 :** Determine where the curve  $y = x^3 - 3x + 1$  is concave upward, and where it is concave downward. Also find the inflection points.

**Solution :**

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Now  $f''(x) = 6x$

Thus  $f''(x) > 0$  when  $x > 0$  and  $f''(x) < 0$  when  $x < 0$ .

The test for concavity then tells us that the curve is concave downward on  $(-\infty, 0)$  and concave upward on  $(0, \infty)$ . Since the curve changes

from concave downward to concave upward when  $x = 0$ , the point  $(0, f(0))$  i.e.,  $(0, 1)$  is a point of inflection.

Note that  $f''(0) = 0$

**Example 5.63 :**

Discuss the curve  $y = x^4 - 4x^3$  with respect to concavity and points of inflection.

**Solution :**

$$f(x) = x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Since  $f''(x) = 0$  when  $x = 0$  or  $2$ , we divide the real line into three intervals.

$(-\infty, 0)$ ,  $(0, 2)$ ,  $(2, \infty)$  and complete the following chart.

**Example 5.64 :** Find the points of inflection and determine the intervals of convexity and concavity of the Gaussian curve  $y = e^{-x^2}$

**Solution:**

$$y' = -2xe^{-x^2}; y'' = 2e^{-x^2}(2x^2 - 1)$$

(The first and second derivatives exist everywhere). Find the values of  $x$  for which  $y'' = 0$

$$2e^{-x^2}(2x^2 - 1) = 0$$

$$x = -\frac{1}{\sqrt{2}}, \text{ or } x = \frac{1}{\sqrt{2}}$$

when  $x < -\frac{1}{\sqrt{2}}$  we have  $y'' > 0$  and when  $x > \frac{1}{\sqrt{2}}$  we have  $y'' < 0$

The second derivative changes sign from positive to negative when passing



through the point  $x = -\frac{1}{\sqrt{2}}$ . Hence, for  $x = -\frac{1}{\sqrt{2}}$ , there is a point of inflection on the curve; its co-ordinates are  $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

When  $x < \frac{1}{\sqrt{2}}$  we have  $y'' < 0$  and when  $x > \frac{1}{\sqrt{2}}$  we have  $y'' > 0$ . Thus there is also a point of inflection on the curve for  $x = \frac{1}{\sqrt{2}}$ ; its co-ordinates are  $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ . (Incidentally, the existence of the second point of inflection follows directly from the symmetry of the curve about the  $y$ -axis). Also from the signs of the second derivatives, it follows that

**Example 5.65 :**

Determine the points of inflection if any, of the function  
 $y = x^3 - 3x + 2$

**Solution :**

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x+1)(x-1)$$

$$\frac{d^2y}{dx^2} = 6x = 0 \Rightarrow x = 0$$

$$\text{Now } \frac{d^2y}{dx^2}(-0.1) = 6(-0.1) < 0 \text{ and}$$

$$\frac{d^2y}{dx^2}(0.1) = 6(0.1) > 0. \text{ In the neighbourhood } (-0.1, 0.1)$$

of 0,  $y''(-0.1)$  and  $y''(0.1)$  are of opposite signs. Therefore  $(0, y(0))$  i.e.,  $(0, 2)$  is a point of inflection.

for  $-\infty < x < -\frac{1}{\sqrt{2}}$  the curve is concave upward ;

for  $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$  the curve is convex upward ;

for  $\frac{1}{\sqrt{2}} < x < \infty$  the curve is concave upward.

**Example 5.66 :**

Test for points of inflection of the curve  $y = \sin x$ ,  $x \in (0, 2\pi)$

**Solution :**  $y' = \cos x$

$$y'' = -\sin x = 0 \Rightarrow x = n\pi, n = 0, \pm 1, \pm 2, \dots$$

since  $x \in (0, 2\pi)$ ,  $x = \pi$  corresponding to  $n = 1$ .

Now  $y'' \text{ ' } (.9\pi) = -\sin (.9\pi) < 0$  and

$y'' \text{ ' } (1.1\pi) = -\sin (1.1\pi) > 0$  since  $\sin (1.1\pi)$  is negative

The second derivative test confirms that  $(\pi, f(\pi)) = (\pi, 0)$  is a point of inflection.

## DIFFERENTIAL CALCULUS-APPLICATION-I

### EXERCISE SUMS:

#### Exercise 5.1

(1) A missile fired from ground level rises  $x$  meters vertically upwards in

$T$  seconds and  $x = 100t - \frac{25}{2}t^2$ . Find

- (i) The initial velocity of the missile,
- (ii) The time when the height of the missile is a maximum
- (iii) The maximum height reached and



(iv) The velocity with which the missile strikes the ground.

Given (known) : upward displacement in time t sec

$$x(t) = x + 100t - \frac{25}{2}t^2$$

(i) Initial velocity =  $\left(\frac{dx}{dt}\right)_{t=0}$  [i.e.,  $\frac{dx}{dt}$ ] at  $t = 0$

$$\text{Now } \frac{dx}{dt} = 100 - 25t$$

$$\text{initial velocity} = 100 - 25 \times 0 = 100\text{m/s}$$

(ii) At the maximum height velocity,  $\frac{dx}{dt} = 0$

$$100 - 25t = 0. \text{ this gives } t = 4\text{sec}$$

(iii) maximum height reached = height at  $t = 4$

$$= x(4)$$

$$= 100 \times 4 - \frac{25}{2} \times 16$$

$$= 400 - 200 = 200$$

Hence the maximum height reached = height at  $t = 4$

(iv) When the missile reaches the ground, the height  $x = 0$

$$\text{This implies } 100t - \frac{25}{2}t^2 = 0$$

This gives the value of  $t = 0$  or  $8$

$t = 0$  is not admissible

Taking  $t = 8\text{sec.}$ , velocity of the missile when it reaches the ground is

$$\left(\frac{dx}{dt}\right)_{t=8} = 100 - 25 \times 8 = -100 \text{ m/s}$$

(2) A particle of unit mass moves so that displacement after  $t$  secs is given by

$x = 3 \cos (2t - 4)$ . Find the acceleration and kinetic energy at the end of 2 secs.

$$\left[ \text{K. E} = \frac{1}{2} m v^2, m \text{ is mass} \right]$$

Given: Displacement of a particle after  $t$  sec,

$$x(t) = 3 \cos (2t - 4)$$

Mass of particle  $m = 1$

$$\text{Velocity } v = \frac{dx}{dt} = -6 \sin(2t - 4)$$

$$\text{Acceleration } a = \frac{d^2x}{dt^2} = -12 \cos (2t - 4)$$

$$\text{Velocity at the end of 2sec.} = \left( \frac{dx}{dt} \right)_{t=2} = -12 \cos(4 - 4) = -12$$

Kinetic energy  $\text{K. E} = \frac{1}{2} \times \text{mass} \times \text{square of velocity}$

$$= \frac{1}{2} m \times v^2$$

$$= \frac{1}{2} \times 1 \times 0 = 0$$

(3) The distance  $x$  meters traveled by a vehicle in time  $t$  seconds after the brakes are applied is given by :  $x = 20t - 5/3t^2$ . Determine

(i) The speed of the vehicle (in km/hr) at the instant the brakes are applied and

(ii) the distance the car travelled before it stops.

given : The distance travelled by the car after the brakes are applied

$$x(t) = 20t - \frac{5}{3}t^2$$

(i) At the instant when the brakes are applied  $t = 0$

$$\therefore \text{required velocity} = \left(\frac{dx}{dt}\right)_{t=0}$$

$$\text{Now } \frac{dx}{dt} = 20 - \frac{10}{3}t = 20 \text{ at } t = 0$$

$\therefore$  The speed of the vehicle when the brakes are applied = 20 m/sec

$$\frac{20 \times 3600}{1000} = 72 \text{ km/hr}$$

(ii) When the vehicle stops velocity = 0

$$\text{This implies } 20 - \frac{10}{3}t = 0$$

$$\text{This gives } t = 6\text{sec}$$

Thus the vehicle stops at the end of 6 seconds

The distance travelled before it stops is

$$x(6) = 20 \times 6 - \frac{5}{3} \times 6^2 = 60\text{m}$$

(4) Newton's law of cooling is given by  $\theta = \theta_0 e^{-kt}$  where the excess of temperature at zero time is  $\theta_0^\circ \text{C}$  and at time  $t$  seconds is  $\theta^\circ \text{C}$ . Determine the rate of change of temperature after 40 s, given that

$$\theta_0 = 16^\circ \text{C} \text{ and } k = -0.03. \quad [e^{1.2} = 3.3201]$$

Given: temperature at  $t$  sec.  $\theta_0 e^{-kt}$

$$\theta_0 = 16^\circ \text{C}$$

$$k = -0.03$$

Rate of change of temperature is

$$\frac{d\theta}{dt} = -k\theta_0 e^{-kt}$$

$$\begin{aligned} \therefore \left( \frac{d\theta}{dt} \right)_{t=40} &= -(-0.03) \times 16 \times e^{-(-0.03) \times 40} \\ &= 0.48 \times e^{1.2} = 0.48 \times 3.3201 = 1.5936^\circ C - s \end{aligned}$$

(5) The altitude of a triangle is increasing at a rate of 1 cm / min while the area of the triangle is increasing at a rate of 2 cm<sup>2</sup>/ min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm<sup>2</sup>.

**Solution** : let b cm and h cm respectively denote the base and altitude of given triangle ABC at time t min . then the area of triangle is

$$\Delta = \frac{1}{2} bh$$

Given:  $\frac{dh}{dt} = 1 \text{ cm / min}$

$$\frac{d\Delta}{dt} = 2 \text{ cm}^2/\text{min. to find } \frac{db}{dt} = ?$$

Initially when h = 10 and Δ = 100 we have

$$100 = \frac{1}{2} \times b \times 10$$

$$\therefore b = 20 \text{ cm}$$

$$\text{Now } \frac{d\Delta}{dt} = \frac{1}{2} \left( b \frac{dh}{dt} + h \frac{db}{dt} \right)$$

$$\therefore h \frac{db}{dt} = 2 \times \frac{d\Delta}{dt} - b \frac{dh}{dt}$$

$$\begin{aligned} \text{Or } \frac{db}{dt} &= \frac{2 d\Delta}{h dt} - \frac{b dh}{h dt} \\ &= \frac{2}{10} \times (2) - \frac{20}{10} \times 1 \\ &= \frac{4}{10} - \frac{20}{10} = -1.6 \end{aligned}$$

The base of the triangle is decreasing at the rate of 1.6 cm / min.

(6) At noon, ship A is 100 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4.00 p.m.?

**solution:**

The given situation is described as in the fig. 5.2 and 5.3

let P and Q be the initial positions of the ships A and B respectively

**After 4 hours,**

Let x be the distance between Q and A, Y be the distance between Q and B and z be the distance between the ships (i.e., AB)

$$\text{Now } z^2 = x^2 + y^2$$

$$2z = \frac{dz}{dt} = 2x \frac{dx}{dt} + \frac{dy}{dt}$$

$$\text{We know that } \frac{dx}{dt} = \text{speed of ship A} = 35$$

$$\frac{dy}{dt} = \text{speed of ship B} = 25$$

$$\text{When } t = 4, x = 40, y = 100$$

$$\text{and } z = \sqrt{x^2 + y^2} = \sqrt{40^2 + 100^2} = 20\sqrt{9}$$

$$(1) \Rightarrow 20 \cdot \sqrt{9} \frac{dz}{dt} = 40 \times 35 + 10 \times 25$$

$$\frac{dz}{dt} = \frac{195}{\sqrt{29}}$$

i.e., the distance between the ships changing at the rate of  $\frac{195}{\sqrt{29}}$  km/hr

(7) Two sides of a triangle are 4m and 5m in length and the angle between them is increasing at a rate of 0.06 rad/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is  $\pi/3$ .

**solution :**

Given triangle be ABC. Then we have  $b = 5\text{cm}$ ,  $c = 4\text{cm}$

Let  $\theta$  be the angle between AB and AC in radians at time  $t$ .

We are also given

$$\frac{d\theta}{dt} =$$

0.6 rad/sec. If  $\Delta$  be the area of triangle at time  $t$ . To find  $\frac{d\Delta}{dt}$  when

$$\theta = \frac{\pi}{3}$$

We know that area of  $\Delta$  ABC

$$\Delta = \frac{1}{2} bc \sin\theta$$

$$\frac{d\Delta}{dt} = \left(\frac{1}{2}bc\right) \cos\theta \frac{d\theta}{dt}$$

$$\begin{aligned} \left(\frac{d\Delta}{dt}\right)_{\theta=\pi/3} &= \frac{1}{2} \times 5 \times 4 \times \cos \frac{\pi}{3} \times (0.06) \\ &= 0.3\text{m}^2/\text{sec}. \end{aligned}$$

(8) Two sides of a triangle have length 12 m and 15 m. The angle between them is increasing at a rate of  $2^\circ/\text{min}$ . How fast is the length of third side increasing when the angle between the sides of fixed length is  $60^\circ$ ?

**solution :**

If AB and BC are the given sides then we have

$$AB = c = 12\text{m and}$$

$$BC = a = 15\text{m}$$

let  $\theta^\circ$  be the angle between AB and BC in time t minutes

$$\text{Then } \frac{d\theta}{dt} = 2^\circ/\text{min} = \frac{\pi}{90}\text{rad/min}$$

To find  $\frac{db}{dt}$  when  $\theta = 60^\circ$

$$\text{We know that } b^2 = c^2 + a^2 - 2ac \cos \theta$$

$$2b \frac{db}{dt} = 0 + 0 - 2ac \left( - \sin \theta \frac{d\theta}{dt} \right)$$

$$\frac{db}{dt} = \frac{ac \sin \theta}{b} \frac{d\theta}{dt}$$

When  $a = 15$ ,  $c = 12$  and  $\theta = 60^\circ$ ,

$$b^2 = 12^2 + 15^2 - 2 \times 15 \times 12 \times \cos 60^\circ$$

$$= 369 - 180$$

$$= 189$$

$$\left( \frac{db}{dt} \right)_{\theta=60^\circ} = \frac{180 \times \frac{\sqrt{3}}{2} \times \frac{\pi}{90}}{\sqrt{189}} \text{m/min}$$

$$= \frac{\pi}{\sqrt{63}} \text{m/min}$$

(9) Gravel is being dumped from a conveyor belt at a rate of 30 ft<sup>3</sup>/min and its coarsened such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

**Solution :**

Let  $r$ ,  $h$  respectively denote the base radius and height of a cone of volume  $V$ , at time  $t$  min. Then we are given that  $2r = h$

To find  $\frac{dh}{dt}$  when  $h = 10$  ft and  $\frac{dv}{dt} = 30$  ft<sup>3</sup>/min.

$$\text{Volume of cone } V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12} (h^3)$$

$$\frac{dv}{dt} = \frac{\pi}{12} \times 3h^2 \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{4 \frac{dv}{dt}}{\pi h^2}$$

$$\left(\frac{dh}{dt}\right) = \frac{4 \times 30}{\pi \times 100} = \frac{12}{10\pi} = \frac{6}{5\pi} \text{ ft/min.}$$

$\therefore$  The height of the cone is increasing at the rate of  $\frac{6}{5\pi}$  ft/min.

## EXERCISE 5.2

1. Find the equation of the tangent and normal to the curves

Solution:

Equation of the tangent and normal at  $(x_1, y_1)$  are

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\text{where } m = \left(\frac{dy}{dx}\right)$$

Given :  $y = x^2 - 4x - 5$  ; at  $x = -2$

$$y = (-2)^2 - 4(-2) - 5 = 7$$

$$\therefore (x_1, y_1) = (-2, 7)$$



$$\frac{dy}{dx} = 2x - 4$$

$$m = \left(\frac{dy}{dx}\right)_{(-2,7)} = 2(-2) - 4 = -8$$

*Equation of the tangent is  $y - 7 = -8(x + 2)$*

$$\text{i.e., } 8x + y + 9 = 0$$

Equation of the normal is  $y - 7 = -\frac{1}{8}(x + 2)$

$$\text{i.e., } -x + 8y - 58 = 0$$

$$\text{or } x - 8y + 58 = 0$$

ii) we have  $y = x - \sin x \cos x$  and  $x = \frac{\pi}{2}$

$$\therefore y = \frac{\pi}{2} - \sin \frac{\pi}{2} \cdot \cos \frac{\pi}{2}$$

$$(x_1, y_1) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

To find the equations of the tangent and normal at  $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$m = \left(\frac{dy}{dx}\right) = 1 - \cos \pi = 1 - (-1) = 2$$

*Equation of the tangent at  $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$  is*

$$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{2}\right)$$

$$\text{or } 2x - y - \frac{\pi}{2} = 0$$

Equation of the normal is  $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$

$$\text{Or } x + 2y - \frac{3\pi}{2} = 0$$

iii)  $y = 2 \sin^2 3x$  and  $x = \frac{\pi}{6}$

$$y = 2 \sin^2 \frac{3\pi}{6} = 2 \text{ when } x = \frac{\pi}{6}$$

$$(x_1, y_1) = \left( \frac{\pi}{6}, 2 \right)$$

$$\frac{dy}{dx} = 6 \sin 6x$$

$$m = \left( \frac{dy}{dx} \right)_{\left( \frac{\pi}{6}, 2 \right)} = 0$$

In this case the given curve has a horizontal tangent with

$y = 2$  (since  $y=y_1$ ) and

The equation normal is

$x = \frac{\pi}{6}$  (since  $x=x_1$ )

(iv)  $y = \frac{1+\sin x}{\cos x}, x = \frac{\pi}{4}$

$$y = \frac{1 + \sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = \sqrt{2} + 1$$

$$\frac{dy}{dx} = \sec x, \tan x + \sec^2 x$$

$$m = \left( \frac{dy}{dx} \right)_{\left( \frac{\pi}{4}, \sqrt{2}+1 \right)} = \sqrt{2} \cdot 1 + 2 = 2 + \sqrt{2}$$

Equation tangent is

$$y - (\sqrt{2} + 1) = (2 + \sqrt{2}) \left(x - \frac{\pi}{4}\right)$$

$$(2 + \sqrt{2})x - y + \left(\sqrt{2} + 1 - (2 + \sqrt{2})\frac{\pi}{4}\right) = 0$$

Equation of normal is

$$y - (\sqrt{2} + 1) = \frac{1}{(2 + \sqrt{2})} \left(x - \frac{\pi}{4}\right)$$

(2) find the point of  $x^2 - y^2 = 2$  at which the slope of the tangent is 2.

Solution :

Given  $x^2 - y^2 = 2$ ; slope  $m = 2$

Let  $(x_1, y_1)$  be the point on the curve such that

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2$$

Now  $2x - 2y \frac{dy}{dx} = 0$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 \Rightarrow \frac{x_1}{y_1} = 2$$

$$x_1 = 2y_1$$

$(x_1, y_1)$  lies on  $x^2 - y^2 = 2$

$$x_1^2 - y_1^2 = 2$$

$$\therefore (2y_1)^2 - y_1^2 = 2$$

$$\text{Or } y_1^2 = \frac{2}{3}$$

$$y_1 = \pm \sqrt{\frac{2}{3}}$$

$$\text{When } y_1 = \sqrt{\frac{2}{3}}, x_1 = 2\sqrt{\frac{2}{3}}$$

$$\text{When } y_1 = -\sqrt{\frac{2}{3}}, x_1 = -2\sqrt{\frac{2}{3}}$$

$$\therefore \text{ The points are } \left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \text{ and } \left(-2\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$$

3. Find at which points on the circle

$$x^2 - y^2 = 13 \text{ the tangent is parallel to the line } 2x + 3y = 7$$

Solution:

$$\text{Given: } x^2 - y^2 = 13$$

Let  $(x_1, y_1)$  be the point such that the tangent is parallel to  $2x + 3y = 7$ .

$$\text{Then } \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{2}{3}$$

$$\text{Now } 2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \frac{x_1}{y_1} = -\frac{x}{y}$$

$$x_1 = \frac{2}{3}y_1$$

This lies on  $x^2 - y^2 = 13$

$$x_1^2 - y_1^2 = 13$$

$$\left(\frac{2}{3}y_1\right)^2 + y_1^2 = 13$$

$$13y_1^2 = 9 \times 13$$

$$y_1 = \pm 3$$

∴ The points are (2,3) and (-2,-3)

4. At what points on the curve  $x^2 + y^2 - 2x - 4y + 1 = 0$  the tangent is parallel to (i)  $x$ -axis, (ii)  $y$ -axis.

Solution:

$$\text{Given: } x^2 + y^2 - 2x - 4y + 1 = 0 \dots (1)$$

Diff. w.r.t  $x$ ,

$$2x + 2y \frac{dy}{dx} - 2 - 4 \frac{dy}{dx} = 0$$

$$(y - 2) \frac{dy}{dx} = 1 - x$$

$$\frac{dy}{dx} = \frac{1 - x}{y - 2}$$

given that tangent is parallel to  $x$ -axis

$$\therefore \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1 - x}{y - 2} = 0 \Rightarrow x = 1$$

When  $x = 1$  (1)  $\Rightarrow y^2 - 4y = 0$

$$\Rightarrow y(y - 4) = 0$$

$$\Rightarrow y = 0 \text{ or } y = 4$$

Therefore the points are (1, 0) and (1, 4)

given that tangent is parallel to  $y$ -axis

$$\therefore \frac{dy}{dx} = 0 \Rightarrow \frac{y - 2}{1 - x} = 0 \Rightarrow y = 2$$

When  $y=2$ , (1)  $\Rightarrow x^2 - 2x - 3 = 0$

$$\Rightarrow (x - 3)(x + 1) = 0$$

$$\Rightarrow x = 3 \text{ or } = -1$$

Therefore the points are (3, 2) and (-1, 2)

5. Find the equations of those tangents to the circle  $x^2 + y^2 = 52$ , when are parallel to the straight line  $2x + 3y = 6$ .

Solution:

Given circle :  $x^2 + y^2 = 52$

Diff.w.r.t.x,

$$2x + 2y \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{-x}{y}$$

Let  $(x_1, y_1)$  be the tangential point.

At  $(x_1, y_1)$ , slope of the tangent is  $m_1 = \frac{dy}{dx} = \frac{-x_1}{y_1} \dots\dots(1)$

Slope of the line  $2x + 3y - 6 = 0$  is  $m_2 = \frac{-a}{b} = \frac{-2}{3} \dots\dots(2)$

Since the tangent is parallel to the line,  $m_1 = m_2$

From (1) & (2)  $\frac{-x_1}{y_1} = \frac{-2}{3}$

$$i.e., y_1 = \frac{3x_1}{2}$$

$$x_1^2 + \frac{9x_1^2}{4} = 52$$

$$13x_1^2 = 208$$

$$\Rightarrow x_1^2 = 16$$

$$\Rightarrow x_1 = \pm 4$$

$$\text{At } x_1 = \pm 4, y_1 = \frac{3(\pm 4)}{2} = \pm 6$$

The points are (4, 6) and (-4, -6)

Therefore slope of tangent is  $m = \frac{-2}{3}$

Equation the tangent at (4, 6) is  $y - 6 = \frac{-2}{3}(x - 4)$

$$3y - 18 = -2x + 8$$

$$2x + 3y - 26 = 0$$

Equation the tangent at (-4, -6) is  $y + 6 = \frac{-2}{3}(x + 4)$

$$3y + 18 = -2x - 8$$

$$2x + 3y + 26 = 0$$

6. Find the equation of a normal to  $y = x^3 - 3x$  that is parallel to

$$2x + 18y - 9 = 0$$

Solution:

given curve:  $y = x^3 - 3x$  .....(1)

Diff.w.r.t.x,  $\frac{dy}{dx} = 3x^2 - 3$

Let  $(x_1, y_1)$  be the tangential point.

At  $(x_1, y_1)$ , slope of the tangent is  $m_1 = \frac{dy}{dx} = 3x_1^2 - 3$  .....(2)

Slope of the line  $2x + 18y - 9 = 0$  is  $m_2 = \frac{-a}{b} = \frac{-1}{9}$  ... .. (3)

Since the tangent is parallel to the line,  $m_1 = m_2$

$$\text{From (1) \& (2) } \frac{-1}{3x_1^2-3} = \frac{-1}{9}$$

$$\Rightarrow 3(x_1^2 - 1) = 9$$

$$x_1^2 - 1 = 3$$

$$x_1^2 = 4$$

$$x_1 = \pm 2$$

When  $x_1 = 2$ ,

$$y_1 = 8 - 6 = 2$$

when  $x_2 = -2$ ,

$$y_2 = -8 + 6 = -2$$

The points are (2, 2) and (-2, -2)

Equation of normal at (2, 2) is  $y - 2 = \frac{-1}{9}(x - 2)$

$$9y - 18 = -x + 2$$

$$\therefore x + 9y - 20 = 0$$

Equation of normal at (-2, -2) is  $y + 2 = \frac{-1}{9}(x + 2)$

$$9y + 18 = -x - 2$$

$$\therefore x + 9y + 20 = 0$$

7. let P be a point on the curve  $y = x^3$  and suppose that tangent line at P



Intersects the curve again at Q. prove that the slope at Q is four times the slope at P.

Solution:

Let P  $(t, t^3)$  be any point on the curve  $y = x^3$

$$\therefore \text{slope at P} = m_1 = 3t^2$$

Equation of tangent at P  $(t, t^3)$  is  $y - y_1 = m(x - x_1)$

$$y - t^3 = 3t^2(x - t) \dots\dots (1)$$

It passes through Q  $(t, t_1^3)$

$$\therefore t_1^3 - t^3 = 3t^2(t_1 - t)$$

$$(t_1 - t)(t_1^2 + t_1t + t^2) = 3t^2(t_1 - t)$$

$$\div (t_1 - t), t_1^2 + t_1t + t^2 = 3t^2$$

$$t_1^2 + t_1t + t^2 = 0$$

$$(t_1 - t)(t_1 + 2t) = 0$$

$$t_1 = -2t \quad [\because t_1 \neq t]$$

$$\text{Slope of tangent at Q} = m_2 = 3t_1^2$$

$$= 3(-2t)^2$$

$$= 4(3t^2)$$

$$m_2 = 4m_1$$

$\therefore$  The slope at Q = 4 (slope at P)

8. prove that the curves  $2x^2 + 4y^2 = 1$  and  $2x^2 + 4y^2 = 1$  cut each other at right angles.

Solution:

We have  $2x^2 + 4y^2 = 1$  and  $2x^2 + 4y^2 = 1$

Let  $(x_1, y_1)$  be the point of intersection.

Now  $(x_1, y_1)$  lies on both the curves

$$2x_1^2 + 4y_1^2 = 1$$

$$6x_1^2 - 12y_1^2 = 1$$

Before equation solving we get,

$$4x_1^2 = 16y_1^2$$

$$\frac{x_1^2}{y_1^2} = 4 \quad \dots\dots (1)$$

Differentiate the two equations.

$$\text{Now } 4x + 8y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$\text{Now } m_1 = \left( \frac{dy}{dx} \right)_{(x_1, y_1)} = -\frac{x_1}{2y_1}$$

$$\text{Now } 12x - 24y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{2y}$$

$$\text{Now } m_2 = \left( \frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{x_1}{2y_1}$$

Therefore of their slopes

$$\begin{aligned} m_1 \cdot m_2 &= \left(-\frac{x_1}{2y_1}\right) \left(\frac{x_1}{2y_1}\right) \\ &= -\frac{x_1}{4y_1} = -\frac{1}{4} \times 4 = -1 \quad [by\ 1] \end{aligned}$$

⇒ The two curves cut at right angles.

9. at what angle  $\theta$  do the curves  $y = a^x$  and  $y = b^x$  intersect ( $a \neq b$ ).

Solution:

given  $y = a^x$  and  $y = b^x$

let  $P(x_1, y_1)$  be the point of intersection of the given curves.

$$\therefore y_1 = a^{x_1} \quad \text{and} \quad y_1 = b^{x_1}$$

$$\Rightarrow a^{x_1} = b^{x_1}$$

$$\Rightarrow x_1 \log a = x_1 \log b$$

$$\Rightarrow x_1 [\log a - \log b] = 0$$

$$\Rightarrow x_1 = 0$$

$$\Rightarrow y_1 = 1$$

$$\therefore (x_1, y_1) = (0, 1)$$

$$y = a^x$$

$$\frac{dy}{dx} = a^x \log a$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$= a^0 \log a$$

$$= \log a$$

$$y = b^x$$

$$\frac{dy}{dx} = b^x \log b$$

$$m_2 = \left( \frac{dy}{dx} \right)_{(x_1, y_1)}$$

$$= b^0 \log b$$

$$= \log b$$

If  $\theta$  be the angle between them then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\log a - \log b}{1 + \log a \log b} \right|$$

$$\theta = \tan^{-1} \left[ \left| \frac{\log a - \log b}{1 + \log a \log b} \right| \right]$$

10. show that the equation of the normal to the curve  $x = a \cos^3 \theta$

$y = a \sin^3 \theta$  at is  $x \cos \theta - y \sin \theta = a \cos 2\theta$ .

Solution:

Take the any point  $\theta$  as  $(a \cos^3 \theta, a \sin^3 \theta)$

At  $\theta$ ,  $x = a \cos^3 \theta \quad \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$

$y = a \sin^3 \theta \quad \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

At  $\theta$ , slope of the tangent is  $m = \frac{dy}{dx}$

$$\begin{aligned} &= \frac{\frac{dx}{d\theta}}{\frac{dy}{d\theta}} \\ &= \frac{-3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} \\ &= \frac{-\sin \theta}{\cos \theta} \end{aligned}$$

Equation of normal at  $\theta (a \cos^3 \theta, a \sin^3 \theta)$  is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - a \sin^3 \theta = -\left(\frac{-\sin \theta}{\cos \theta}\right)(x - a \cos^3 \theta)$$

$$\Rightarrow y \sin \theta - a \sin^4 \theta = x \cos \theta - a \cos^4 \theta$$

$$\Rightarrow x \cos \theta - y \sin \theta + a(\sin^4 \theta - \cos^4 \theta) = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta + a(\sin^2 \theta - \cos^2 \theta) = 0$$

$$x \cos \theta - y \sin \theta = a \cos 2\theta.$$

11. if the curve  $y^2=x$  and  $xy=k$  are orthogonal then prove that  $8k^2=1$

Solution:

$$y^2=x; xy=k$$

Solving these two equations we have the point of intersection as

$$(k^{2/3}, k^{1/3})$$

$$y^2=x \Rightarrow 2y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

i.e., at  $(k^{2/3}, k^{1/3})$ ,  $m_1 = \frac{1}{2k^{1/3}}$

$$xy = k \Rightarrow y = \frac{k}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{k}{x^2}$$

At  $(k^{2/3}, k^{1/3})$ ,  $m_2 = -\frac{1}{k^{1/3}}$

But they are orthogonal and therefore and therefore  $m_1 m_2 = -1$

$$\Rightarrow \frac{1}{2k^{1/3}} \cdot \left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow \frac{1}{2k^{2/3}} = 1$$

$$2k^{2/3} = 1$$

$$\Rightarrow 8k^2 = 1.$$

### Exercise 5.3

1. verify Rolle's theorem for the following functions :

(i)  $f(x) = \sin x, 0 \leq x \leq \pi$

(ii)  $f(x) = x^2, 0 \leq x \leq 1$

(iii)  $f(x) = |x - 1|, 0 \leq x \leq 2$

(iv)  $f(x) = 4x^3 - 9x, -\frac{3}{2} \leq x \leq \frac{3}{2}$

Solution:

(i)  $f(x) = \sin x, 0 \leq x \leq \pi$

We have  $f(x) = \sin x$ ,  $0 \leq x \leq \pi$ .  $f(x)$  is continuous on  $[0, \pi]$  and Differentiable on  $(0, \pi)$ . Also,  $f(0) = f(\pi)$ . Thus  $f$  satisfies all the conditions of Rolles Theorem.

Therefore, there exists  $c \in (0, \pi)$  such that  $f'(c) = 0$ ,

$$\text{Now } f'(x) = \cos x$$

$$\therefore f'(c) = 0$$

$$\Rightarrow \cos c = 0$$

$$\Rightarrow c = (2n + 1)\frac{\pi}{2}, n \in Z$$

But the  $c$  that lies in  $(0, \pi)$  is  $\frac{\pi}{2}$  which corresponds to  $n=0$ . Therefore the suitable  $c$  of Rolles theorem is  $\frac{\pi}{2}$ .

$$(ii) f(x) = x^2, \quad 0 \leq x \leq 1$$

$f(x)$  being a polynomial function, it is continuous on  $[0, 1]$  and differentiable in  $(0, 1)$ .

but  $f(0) \neq f(1)$ .

Therefore Rolles theorem is not applicable to the given function.

$$(iii) f(x) = |x - 1|, 0 \leq x \leq 2$$

This function is continuous on  $[0, 2]$  but not differentiable at  $x=1 \in (0, 2)$ . However  $f$  satisfies  $f(0) = 1 = f(2)$ . Because,  $f$  fails to satisfy the differentiability condition of Rolles theorem, it cannot be applied to the given function.

$$(iv) f(x) = 4x^3 - 9x, \quad -\frac{3}{2} \leq x \leq \frac{3}{2}$$

$f(x)$  being a polynomial function, it is continuous on  $\left[-\frac{3}{2}, \frac{3}{2}\right]$  and differentiable on  $\left(-\frac{3}{2}, \frac{3}{2}\right)$ . And  $f\left(-\frac{3}{2}\right) = 0 = f\left(\frac{3}{2}\right)$ . Thus  $f$  satisfies all the three conditions of Rolles theorem and hence there exist a  $c \in \left(-\frac{3}{2}, \frac{3}{2}\right)$  satisfying  $f'(c) = 0$

$$\text{now } f'(x) = 12x^2 - 9$$

$$\therefore f'(c) = 0$$

$$\Rightarrow 12c^2 - 9 = 0$$

$$\Rightarrow c^2 = \frac{3}{4}$$

$$\therefore c = \pm \frac{\sqrt{3}}{2}$$

Note that both  $-\frac{\sqrt{3}}{2}$  and  $\frac{\sqrt{3}}{2}$  lie in  $\left(-\frac{3}{2}, \frac{3}{2}\right)$ . Therefore both  $-\frac{\sqrt{3}}{2}$  and  $\frac{\sqrt{3}}{2}$

Are the suitable values of  $c$ .

2. using Rolles theorem find the points on the curve  $y = x^2 + 1$ ,  $-2 \leq x \leq 2$  where the tangent is parallel to  $x$ -axis.

Solution:

$$y = x^2 + 1, \quad -2 \leq x \leq 2$$

$y = f(x) = x^2 + 1$  being a polynomial function.

It is continuous On  $[-2, 2]$  and differentiable on  $(-2, 2)$ .

Now  $f(-2) = 5 = f(2)$ .



Since  $f(x)$  satisfies all the three conditions of Rolles theorem, there exist a 'c' satisfying  $f'(c) = 0$ . That is the tangent to the curve is parallel to the  $x$ - axis.

$$\text{Now } f(x) = x^2 + 1$$

$$f'(x) = 2x$$

$$\therefore f'(c) = 0 \Rightarrow 2c = 0$$

$$\Rightarrow C = 0 \in [-2, 2]$$

When  $c = 0$ ,  $f(0) = 1$

*$\therefore$  the tangent to the curve*

*$y = x^2 + 1$  is parallel to the  $x$  – axis at  $(0, 1)$*

#### Exercise 5.4

1. Verify Lagrange's law of mean for the following functions:

(i)  $f(x) = 1 - x^2, [0, 3]$

Solution:

We have  $f(x) = 1 - x^2, [0, 3]$

$f(x)$  being a polynomial, it is continuous in  $[0, 3]$  and differentiable on  $(0,3)$ , therefore by the law of the mean, there exists at least one point  $c$  belongs to  $(0, 3)$  such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\text{i.e., } -2c = \frac{f(3)-f(0)}{3-0} = \frac{-8-1}{3} = -3$$

$$\therefore c = \frac{3}{2} \in (0, 3)$$

This verifies LaGrange's law of the mean.

$$(ii) f(x) = \frac{1}{x}, [1, 2]$$

Solution:

$f(x) = \frac{1}{x}$  is continuous on  $[1, 2]$  and differentiable on

$(1, 2)$  Therefore by LaGrange's law of the mean,

there exists at least one  $c \in (1, 2)$  satisfying

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\text{i.e., } -\frac{1}{c^2} = -\frac{1}{2}$$

$$\text{or } c^2 = 2 \text{ or } c = \pm\sqrt{2}$$

Clearly  $-\sqrt{2} \notin (1, 2)$  but  $\sqrt{2} \in (1, 2)$ , therefore the suitable value of  $c$  is  $\sqrt{2}$ .

$$(iii) f(x) = 2x^3 + x^2 - x - 1, [0, 2]$$

$f(x)$  being polynomial, it is continuous on  $[0, 2]$  and differentiable on  $(0, 2)$

$$f'(x) = 6x^2 + 2x - 1$$

by LaGrange's law of the mean, there exists at least one  $c \in (0, 2)$  such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{17 + 1}{2} = 9$$

$$\text{i.e., } 6c^2 + 2c - 1 = 9$$

$$\text{or } 6c^2 + 2c - 10 = 0$$

$$3c^2 + c - 5 = 0$$

$$\Rightarrow c = \frac{-1 \pm \sqrt{61}}{6}$$

$$\Rightarrow \text{now } c = \frac{-1 + \sqrt{61}}{6} \in (0, 2) \text{ and } c = \frac{-1 - \sqrt{61}}{6} \notin (0, 2).$$

This verifies Lagrange's law of the mean with  $c = \frac{-1 + \sqrt{61}}{6}$

$$\text{(iv) } f(x) = x^{\frac{2}{3}}, [-2, 2]$$

$f(x)$  is continuous on  $[-2, 2]$  but  $f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$

which does not exist for  $x = 0$

since  $f(x)$  is not differentiable in  $(-2, 2)$  and in this case Lagrange's law of the mean is not applicable and hence cannot be verified.

$$\text{(v) } f(x) = x^3 - 5x^2 - 3x, (1, 3)$$

$f(x)$  being polynomial, it is continuous on  $(1, 3)$ .

Therefore by Lagrange's law of the mean, there exists at least one  $c$  belongs to  $(1, 3)$  such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 + 7}{2} = -10$$

$$\text{i.e., } 3c^2 - 10c - 3 = 10$$

$$3c^2 - 10c + 7 = 10$$

$$(c - 1)(3c - 7) = 0$$

$$c = 1 \text{ or } c = 7/3$$

now  $1 \notin (1, 3)$  but  $c = 7/3 \in (1, 3)$ .

This verifies the LaGrange's law of mean with  $c = 7/3$

(2) If  $f(1) = 10$  and  $f'(x) \geq 2$  for  $1 \leq x \leq 4$  how small can  $f(4)$  possibly be?

Solution:

We have  $f(1) = 10$  and  $f'(x) \geq 2, 1 \leq x \leq 4$

The condition  $f'(x) \geq 2$  reveals that  $f'(x)$  exists for all  $x \in (1, 4)$  and hence  $f(x)$  is differentiable on  $(1, 4)$  and continuous on  $[1, 4]$ .

There we can apply Lagrange's law of the mean on  $[1, 4]$ . Hence there exists  $c$  belongs to  $(1, 4)$  such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{4 - 1} \quad (\text{since } f(1) = 10)$$

$$f(4) = 3f'(c) + 10$$

$$\Rightarrow f(4) \geq 6 + 10$$

i.e.,  $f(4) \geq 16$ . This says that  $f(4)$  must be at least 16.

(3) At 2.00 pm a car's speedometer reads 30 miles/hr., at 2.10 pm it reads 50 miles/hr. show that sometime between 2.00 and 2.10 the acceleration is exactly 120 miles/hr<sup>2</sup>.

Solution:

Let  $V$  be the velocity reading shown in the speedometer at any time  $t$ . Then the velocity function .

V is a continuous function of  $t$  on  $(2, 2.10)$  and is differentiable on  $(2, 2.10)$

$$\text{Now } V(2) = 30 \text{ miles/hr.}$$

$$V(2.10) = 50 \text{ miles/hr.}$$

By LaGrange's law of the mean there exists  $c$  belongs to  $(2, 2.10)$  such that

$$V'(c) = \frac{V(2.10) - V(2)}{2.10 - 2}$$

$$V'(c) = \frac{50 - 30}{10/60} \text{ miles/hr.}$$
$$= 120 \text{ miles/hr.}$$

Note that  $V$  is velocity and  $V'(t)$  is the acceleration at  $c$  in between 2pm and 2.10 pm.

### Exercise 5.5

1. Obtain the McLaurin's series expansion for :

(i) Let  $f(x) = e^{2x}$

$$\begin{array}{lll} f(x) = e^{2x}; & & f(0) = 1 \\ f'(x) = 2e^{2x}; & & f'(0) = 2 \\ f''(x) = 2^2 e^{2x}; & & f''(0) = 2^2 \\ f'''(x) = 2^3 e^{2x}; & & f'''(0) = 2^3 \end{array}$$

The McLaurin's series expansion for  $e^{2x}$  is

$$e^{2x} = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$
$$= 1 + \frac{2x}{1!} + \frac{2^2}{2!} x^2 + \frac{2^3}{3!} x^3 + \dots$$

$$= 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots$$

(ii) Let  $\cos^2 x$

$$\begin{aligned} f(x) &= \cos^2 x ; & f(0) &= 1 \\ f'(x) &= -\sin 2x; & f'(0) &= 0 \\ f''(x) &= 2 \cos 2x; & f''(0) &= -2 \\ f'''(x) &= +4 \sin 2x; & f'''(0) &= 0 \\ f^{iv}(x) &= 8 \cos 2x ; & f^{iv}(0) &= 8 \end{aligned}$$

$$\cos^2 x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots\dots$$

$$= 1 - \frac{2x^2}{2!} + \frac{x^4}{4!} (8) + \dots$$

$$= 1 - x^2 + \frac{x^4}{3} + \dots$$

(iii) let  $f(x) = \frac{1}{1+x}$  ;

$$f(x) = \frac{1}{1+x} ; \quad f(0) = 1$$

$$f'(x) = -\frac{1}{(1+x)^2} ; \quad f'(0) = -1$$

$$f''(x) = \frac{2}{(1+x)^3} ; \quad f''(0) = 2$$

$$f'''(x) = -\frac{6}{(1+x)^4} ; \quad f'''(0) = -6$$

$$\dots \quad \vdots \quad \vdots$$

$$\begin{aligned} & \dots \vdots \quad \quad \quad \vdots \\ \therefore \frac{1}{1+x} &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots \\ &= 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \dots \\ &= 1 - x + x^2 - x^3 + \dots \end{aligned}$$

(iv) lettan  $x$

$$f(x) = \tan x ; \quad f(0) = 0$$

$$f'(x) = \sec^2 x ; \quad f'(0) = 1$$

$$f''(x) = 1 + \tan^2 x = 1 + f^2(x)$$

$$f'''(x) = 2f(x) \cdot f''(x); \quad f'''(0) = f(0)f''(0) = 0$$

$$f^{(4)}(x) = 2[f(x) \cdot f'''(x) + f'(x) \cdot f''(x)]$$

$$f^{(4)}(0) = 2[f(0) \cdot f'''(0) + f'(0) \cdot f''(0)] = 2$$

i.e.,  $f^{(4)}(0) = 2$

$$f^{(5)}(x) = 2[f(x) \cdot f^{(4)}(x) + f''(x) \cdot f'(x) + 2f'(x)f''(x)]$$

$$f^{(5)}(0) = 2[f(0) \cdot f^{(4)}(0) + f''(0) \cdot f'(0) + 2f'(0)f''(0)]$$

$$\therefore \tan x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

$$= 0 + x + 0 + \frac{x^3}{3!} (2) + 0 + \dots$$

$$\tan x = x + \frac{x^3}{3} + \dots$$

Exercise -5.6

Evaluate the limit for the following if exists.

Solution:

$$(1) \lim_{x \rightarrow 2} \frac{\sin \pi x}{2-x} \left( \frac{0}{0} \text{ form} \right)$$

By l' Hospital's rule ,

$$\lim_{x \rightarrow 2} \frac{\pi \cos \pi x}{-1} = \frac{\pi \cos 2\pi}{-1} = -\pi$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} \left( \frac{0}{0} \text{ form} \right)$$

By l' Hospital's rule,

$$= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x} \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{\sin x}$$

$$= \lim_{x \rightarrow 0} (2 \sec^3 x) = (2)(1) = 2$$

$$(3) \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} \left( \frac{0}{0} \text{ form} \right)$$

Applying l's hospital's form rule



$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1-x^2}} = 1$$

(4)  $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2}$  ( $\frac{0}{0}$  form) Applying l' Hospital' srule

$$\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = \lim_{x \rightarrow 2} \frac{nx^{n-1}}{1} = n2^{n-1}$$

(5)  $\lim_{x \rightarrow \infty} \frac{\sin \frac{2}{x}}{1/x}$

Put  $\frac{1}{x} = y$  then  $y \rightarrow 0$  as  $x \rightarrow \infty$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin \left(\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{y \rightarrow 0} \frac{\sin 2y}{y} \left(\frac{0}{0} \text{ form}\right) \text{ applying l' Hospital' s rule}$$

$$= \lim_{y \rightarrow 0} \left(\frac{2 \cos 2y}{1}\right) = 2(1) = 2$$

(6) put  $\frac{1}{x} = y$

Now  $y \rightarrow 0$  as  $x \rightarrow \infty$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} - 2 \tan^{-1} \left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y \rightarrow 0} \frac{y^2 - 2 \tan^{-1}(y)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{2y - \frac{2}{1+y^2}}{1}$$

(applying l' Hospital's rule)

$$= 0 - \frac{2}{1} = -2$$

(7)  $\lim_{x \rightarrow \infty} \frac{\log_e x}{x}$  ( $\frac{0}{\infty}$  form) applying l' Hospital's rule

$$\lim_{x \rightarrow \infty} \frac{\log_e x}{x} = \lim_{x \rightarrow \infty} \left( \frac{1/x}{1} \right) = 0$$

(8)  $\lim_{x \rightarrow 0} \frac{\cot x}{\cot 2x}$  ( $\frac{\infty}{\infty}$ )

$$\lim_{x \rightarrow 0} \frac{\cot x}{\cot 2x} = \lim_{x \rightarrow 0} \frac{\tan 2x}{\tan x} \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 2x}{\sec^2 x} = \frac{2 \cdot 1}{1} = 2$$

(9)  $\lim_{x \rightarrow 0^+} x^2 \log_e x$

$$= \lim_{x \rightarrow 0^+} \left( \frac{\log_e x}{\frac{1}{x^2}} \right) \left( \frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\left( -\frac{2}{x^3} \right)} = \lim_{x \rightarrow 0^+} \left( \frac{x^3}{-2x} \right) = \lim_{x \rightarrow 0^+} \left( -\frac{x^2}{2} \right) = 0$$

$$(10) \lim_{x \rightarrow 1} x^{\frac{1}{x-1}} (1^\infty \text{ form})$$

$$\text{Let } y = x^{\frac{1}{x-1}}$$

$$\log y = \frac{1}{(x-1)} \log_e x$$

$$\lim_{x \rightarrow 1} \log y = \lim_{x \rightarrow 1} \frac{\log_e x}{x-1} \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = 1$$

$$\text{i.e., } \lim_{x \rightarrow 1} \log y = 1$$

By composite function theorem, we have

$$1 = \lim_{x \rightarrow 1} \log y = \log \left( \lim_{x \rightarrow 1} y \right) = \log \left( \lim_{x \rightarrow 1} x^{\frac{1}{x-1}} \right)$$

$$\Rightarrow \lim_{x \rightarrow 1} x^{\frac{1}{x-1}} = e$$

$$(11) \lim_{x \rightarrow \pi/2} (\tan x)^{\cos x} (\infty^0 \text{ form})$$

$$\text{Let } y = (\tan x)^{\cos x}$$

$$\log y = \cos x \log(\tan x)$$

$$\lim_{x \rightarrow \pi/2} (\log y) = \lim_{x \rightarrow \pi/2} (\cos x) \log(\tan x)$$

$$= \lim_{x \rightarrow \pi/2} \frac{\log(\tan x)}{\sec x} \left( \frac{\infty}{\infty} \text{ form} \right)$$

$$\begin{aligned}
&= \lim_{x \rightarrow \pi/2^-} \left( \frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x} \right) \\
&= \lim_{x \rightarrow \pi/2^-} \left( \frac{\sec x}{\tan^2 x} \right) \\
&= \lim_{x \rightarrow \pi/2^-} (\cos x \cdot \operatorname{cosec}^2 x) = 0.1 = 0
\end{aligned}$$

By composite function theorem,

$$\begin{aligned}
0 &= \lim_{x \rightarrow \pi/2^-} (\log y) = \log \left( \lim_{x \rightarrow \pi/2^-} y \right) \\
\therefore \lim_{x \rightarrow \pi/2^-} y &= e^0 = 1
\end{aligned}$$

(12)  $\lim_{x \rightarrow 0^+} x^x$  ( $0^0$  form)

$$\text{Let } y = x^x$$

$$\log y = x \log x$$

$$\lim_{x \rightarrow 0^+} \log y = \lim_{x \rightarrow 0^+} x \log x \quad (0 \times -\infty \text{ form})$$

$$= \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}} \quad \left( \frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0$$

By composite function theorem,

$$0 = \lim_{x \rightarrow 0^+} \log y = \log \left( \lim_{x \rightarrow 0^+} y \right)$$

$$\therefore \lim_{x \rightarrow 0^+} x^x = 1$$

$$(13) \lim_{x \rightarrow 0} (\cos x)^{1/x} \text{ (} 1^\infty \text{ form)}$$

$$\text{let } y = (\cos x)^{1/x}$$

$$\log y = \frac{1}{x} \log(\cos x)$$

$$\lim_{x \rightarrow 0} (\log y) = \lim_{x \rightarrow 0} \frac{\log \cos x}{x} \text{ (} \frac{0}{0} \text{ form)}$$

$$= \lim_{x \rightarrow 0} \frac{-\tan x}{1} \text{ (Applying l' hospital's rule)}$$

$$= 0$$

By composite function theorem,

$$0 = \lim_{x \rightarrow 0} (\log y) = \log \left( \lim_{x \rightarrow 0} y \right)$$

$$\therefore \lim_{x \rightarrow 0} y = e^0 = 1$$

$$\text{i.e., } \lim_{x \rightarrow 0} (\cos x)^{1/x} = 1$$

### Exercise - 5.7

1. prove that  $e^x$  is strictly increasing function on  $\mathbb{R}$ .

solution:

$$\text{Let } f(x) = e^x$$

$$f'(x) = e^x > 0 \Rightarrow 0 \text{ for all } x \in \mathbb{R}$$

$\therefore f(x) = e^x$  is strictly increasing on  $R$

2. Prove that  $\log x$  is strictly increasing function on  $(0, \infty)$

solution :

$$\text{Let } f(x) = \log x, x \in (0, \infty)$$

$$\text{Now } f'(x) = \frac{1}{x} > 0$$

$f(x)$  is strictly increasing on  $(0, \infty)$

3. which of the following functions are increasing or decreasing on the interval given?

$$\text{Let } f(x) = x^2 - 1, 0 \leq x \leq 2$$

$$f'(x) = 2x \geq 0 \text{ for } 0 \leq x \leq 2$$

$\therefore f(x) = x^2 - 1$  is increasing on  $(0, 2)$

(i)  $\text{Let } f(x) = 2x^2 + 3x; \left[-\frac{1}{2}, -\frac{1}{2}\right]$

$$f'(x) = 4x + 3 > 0 \text{ for } -\frac{1}{2} \leq x \leq \frac{1}{2}$$

$\therefore f(x) = 2x^2 + 3x$  is strictly increasing on  $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(ii)  $\text{Let } f(x) = e^{-x}, 0 < x < 1$

$$f'(x) = e^{-x} < 0, 0 \leq x \leq 1$$

$f'(x) = e^{-x}$  is strictly decreasing on  $[0, 1]$

(iii) Let  $f(x) = x(x-1)(x+1)$ ,  $[-2, -1]$

$$f'(x) = (x-1)(x+1) + x(x+1) + x(x-1)$$

$$= x^2 - 1 + x^2 + x + x^2 - x$$

$$= 3x^2 - 1 > 0 \text{ for } -2 \leq x \leq -1$$

$\therefore f(x) = x(x-1)(x+1)$  is strictly increasing on  $[-2, -1]$

(iv) let  $f(x) = x \sin x$ ;  $0 \leq x \leq \frac{\pi}{4}$

$$f'(x) = \cos x + \sin x$$

$$\therefore f'(x) \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{4}$$

$$\therefore f(x) = x \sin x \text{ is increasing on } \left[0, \frac{\pi}{4}\right]$$

4. Prove that the following function are not monotonic in the intervals given.

(i) Let  $f(x) = 2x^2 + x - 5$ ,  $-1 \leq x \leq 0$

$$f'(x) = 4x + 1 = 0 \text{ if } x = \frac{-1}{4}$$

$$\text{Say } x = \frac{-1}{2}$$

$$\text{Now } f'\left(\frac{-1}{2}\right) = -1 < 0$$

i.e.,  $f(x)$  is decreasing on  $\left[-1, \frac{-1}{4}\right]$

Take any value between  $\frac{-1}{4}$  and 0

$$\text{Say } x = \frac{-1}{8}$$

$$f' \left( \frac{-1}{8} \right) > 0$$

$\therefore f(x)$  is increasing on  $\left[ \frac{-1}{4}, 0 \right]$

Hence the function is not monotonic on  $[-1, 0]$

(ii) Let  $f(x) = x(x-1)(x+1), 0 \leq x \leq 2$

$$\therefore f'(x) < 0 \text{ if } \frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

$$f'(x) = 3x^2 - 1 = 0$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

Take any value between 0 and  $\frac{1}{\sqrt{3}}$  say  $x = \frac{1}{2}$

$$f' \left( \frac{1}{2} \right) = \frac{3}{4} - 1 < 0$$

$$\therefore f'(x) \leq 0 \text{ if } 0 \leq x \leq \frac{1}{\sqrt{3}}$$

$\therefore f$  is increasing on  $\left[ 0, \frac{1}{\sqrt{3}} \right]$



$$f'(x) > 0 \text{ if } \frac{1}{\sqrt{3}} \leq x \leq 2$$

Thus  $f(x)$  is decreasing on  $\left[0, \frac{1}{\sqrt{3}}\right]$  and

increasing on  $\left[\frac{1}{\sqrt{3}}, 2\right]$

Hence  $f(x)$  is not monotonic on  $[0, 2]$

(iii) Let  $f(x) = x \sin x$ ,  $0 < x < \pi$

$$f'(x) = x \cos x + \sin x$$

Now  $f$  is not monotonic if  $f'(x)$  has different signs at different points. So let us check the signs at  $0, \frac{\pi}{2}$  and  $\pi$

$$f'(0) = 0 \cdot 1 + 0 = 0$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} = \frac{\pi}{2} \cdot 0 + \sin \frac{\pi}{2} > 0$$

$$f'(\pi) = \pi \cos \pi + \sin \pi = -\pi + 0 = -\pi < 0$$

Now  $f$  is not monotonic on  $[0, \pi]$

(iv)  $f(x) = \tan x + \cot x$ ,  $0 < x < \frac{\pi}{2}$

$$f'\left(\frac{\pi}{3}\right) = 4 - \frac{4}{3} > 0$$

$$f' \left( \frac{\pi}{6} \right) = \frac{4}{3} - 4 < 0$$

since  $f'$  has different sign at different points,

$f$  is not a monotonic on  $\left(0, \frac{\pi}{2}\right)$

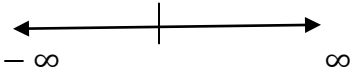
5. Find the intervals on which  $f$  is increasing or decreasing.

(i)  $f(x) = 20 - x - x^2$

$$f'(x) = -1 - 2x ; f'(x) = 0 \Rightarrow x = -\frac{1}{2}$$

Plot this point on the real line

We have two intervals  $-\infty \quad -\frac{1}{2} \quad \infty$



$\left(-\infty, -\frac{1}{2}\right)$  and  $\left(-\frac{1}{2}, \infty\right)$

Interval	$-(1+2x)$	$f'(x)$	interval of strictly <i>inc/dec</i>	Interval of <i>inc /dec</i>
$\left(-\infty, -\frac{1}{2}\right)$	+	$>0$	$\left(-\infty, -\frac{1}{2}\right)$ St. increasing	$\left(-\infty, -\frac{1}{2}\right]$ Increasing
$\left(-\frac{1}{2}, \infty\right)$	-	$<0$	$\left(-\frac{1}{2}, \infty\right)$ St. decreasing	$\left[-\frac{1}{2}, \infty\right)$ decreasing

(ii)  $f(x) = x^3 - 3x + 1$

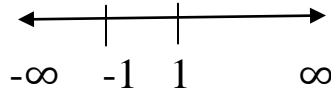
$$f'(x) = 3x^2 - 3$$

$$f'(x) = 0 \Rightarrow x = \pm 1$$

Plot this point on the real line

We have three intervals

$(-\infty, -1), (-1, 1), (1, \infty)$



<i>Interval</i>	$-(3x^2-3)$	$f'(x)$	<i>interval of strictly inc/dec</i>	<i>Interval of inc /dec</i>
$(-\infty, -1)$	+	$>0$	$(-\infty, -1)$ St. increasing	$(-\infty, -1]$ Increasing
$(-1, 1)$	-	$<0$	$(-1, 1)$ St. decreasing	$[-1, 1]$ decreasing
$(1, \infty)$	+	$>0$	$(1, \infty)$ St. increasing	$[1, \infty)$ increasing

(iii)  $f(x) = x^3 + x + 1$

$$f'(x) = 3x^2 + 1 > 0 \text{ for all } x \in \mathbb{R}$$

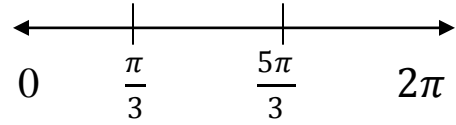
$\therefore f(x)$  is st. increasing on  $\mathbb{R}$ .

(iv)  $f(x) = x - 2 \sin x ; [0, 2\pi]$

$$f'(x) = 1 - 2 \cos x ; f'(x) = 0 \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

These two points divide the interval  $[0, 2\pi]$

Into  $\left[0, \frac{\pi}{3}\right), \left(\frac{\pi}{3}, \frac{5\pi}{3}\right), \left(\frac{5\pi}{3}, 2\pi\right]$



<i>Interval</i>	$1-2\cos x$	$f'(x)$	<i>interval of strictly inc/dec</i>	<i>Interval of inc /dec</i>
$\left[0, \frac{\pi}{3}\right)$	-	$<0$	$\left[0, \frac{\pi}{3}\right)$ St. decreasing	$\left[0, \frac{\pi}{3}\right]$ decreasing
$\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$	+	$>0$	$\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ St. increasing	$\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ increasing
$\left(\frac{5\pi}{3}, 2\pi\right]$	-	$<0$	$\left(\frac{5\pi}{3}, 2\pi\right]$ St. decreasing	$\left[\frac{5\pi}{3}, 2\pi\right]$ decreasing

(v)  $f(x) = x + \cos x, 0 \leq x \leq \pi$

$$f'(x) = 1 - \sin x; f'(x) = 0 \Rightarrow 1 - \sin x = 0 \quad \therefore x = \frac{\pi}{2}$$

$x = \frac{\pi}{2}$  divides the interval  $[0, \pi]$  into  $\left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$

<i>Interval</i>	$1-\sin x$	$f'(x)$	<i>interval of strictly inc/dec</i>	<i>Interval of inc /dec</i>
-----------------	------------	---------	-------------------------------------	-----------------------------

$[0, \frac{\pi}{2})$	+	$>0$	St. increasing on $[0, \frac{\pi}{2})$ ,	$[0, \frac{\pi}{2}]$ Increasing
$(\frac{\pi}{2}, \pi]$	+	$>0$	$(\frac{\pi}{2}, \pi]$ St. increasing	$[\frac{\pi}{2}, \pi]$ increasing

(vi)  $f(x) = \sin^4 x + \cos^4 x ; [0, \frac{\pi}{2}]$

$$\begin{aligned}
 f'(x) &= 4 \sin^3 x \cos x - 4 \cos^3 x \sin x \\
 &= 4 \sin x \cos x (\sin^2 x - \cos^2 x) \\
 &= 2. \sin 2x (-\cos 2x) \\
 &= -\sin 4x
 \end{aligned}$$

$$f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = 0 \Rightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}$$

The values of  $x$  divide the interval  $[0, \frac{\pi}{2}]$  into  $(0, \frac{\pi}{4})$ ,  $(\frac{\pi}{4}, \frac{\pi}{2})$

<i>Interval</i>	$-\sin 4x$	$f'(x)$	<i>interval of strictly inc/dec</i>	<i>Interval of inc /dec</i>
$(0, \frac{\pi}{4})$	-	$<0$	$(0, \frac{\pi}{4})$	$[0, \frac{\pi}{4}]$

			St. decreasing	decreasing
$(\frac{\pi}{4}, \frac{\pi}{2})$	+	$>0$	$(\frac{\pi}{4}, \frac{\pi}{2})$	$[\frac{\pi}{4}, \frac{\pi}{2}]$
			St. increasing	increasing

### Exercise 5.8

1. prove the following inequalities :

i)  $\cos x > 1 - \frac{x^2}{2}, x > 0$

solution:

$$\text{let } f(x) = \cos x - 1 + \frac{x^2}{2}, x > 0$$

$$f'(x) = -\sin x + \frac{2x}{2} = -\sin x + x > 0 \text{ for all } x > 0$$

$\therefore f(x)$  is strictly increasing for all  $x > 0$

$$f(x) > f(0) \text{ for } x > 0$$

$$\left(\cos x - 1 + \frac{x^2}{2}\right) > 1 - 1 + 0$$

$$\left(\cos x - 1 + \frac{x^2}{2}\right) > 0 \Rightarrow \cos x > 1 - \frac{x^2}{2}$$

ii)  $\sin x > x - \frac{x^3}{6}, x > 0$

$$\text{let } f(x) = \sin x - x + \frac{x^3}{6}, x > 0$$

$$f'(x) = \cos x - 1 + 3 \frac{x^2}{6}$$

$$= \cos x - 1 + \frac{x^2}{6} > 0 \text{ for } x > 0$$

$\therefore f(x)$  is strictly increasing on  $x > 0$

$\therefore f(x) > f(0)$  for  $x > 0$

$$\left( \sin x - x + \frac{x^3}{6} \right) > \sin 0 - 0 + 0 \text{ for } x > 0$$

$$\sin x - x + \frac{x^3}{6} > 0$$

$$\text{i.e., } \sin x > x - \frac{x^3}{6}$$

iii)  $\tan^{-1} x < x$  for all  $x > 0$

let  $f(x) = \tan^{-1} x - x$

$$f'(x) = \frac{1}{1+x^2} - 1 = -\frac{x^2}{1+x^2} < 0 \text{ for } x > 0$$

$\therefore f(x)$  is strictly decreasing on  $x > 0$

$f(x) < f(0)$  for  $x > 0$

$$(\tan^{-1} x - x) < (\tan^{-1} 0 - 0)$$

$$\tan^{-1} x - x < 0 \text{ for } x > 0$$

$$\tan^{-1} x < x \text{ for } x > 0$$

iv)  $\log(1+x) < x$  for all  $x > 0$

let  $f(x) = \log(1+x) - x$

$$f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0 \text{ for all } x > 0$$

$\therefore f(x)$  is strictly decreasing on  $x > 0$

$$f(x) < f(0) \text{ for } x > 0$$

$$\text{i.e., } [\log(1+x) - x] < [\log(1+0) - 0]$$

$$\log(1+x) - x < 0$$

$$\log(1+x) < x \text{ for all } x > 0$$

### Exercise 5.9

1. Find the critical numbers and stationary points of each of the following functions.

Solutions:

$$\text{i) } f(x) = 2x - 3x^2$$

$$\text{we have } f(x) = 2x - 3x^2$$

$$f'(x) = 2 - 6x$$

$$f\left(\frac{1}{3}\right) = 2 \times \frac{1}{3} - 3 \times \frac{1}{9}$$

$$= \frac{1}{3}$$

$\therefore x = \frac{1}{3}$  is the critical number and  $\left(\frac{1}{3}, \frac{1}{3}\right)$  is the stationary point.

$$\text{ii) } f(x) = x^3 - 3x + 1$$

$$\text{We have } f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3 = 0 \text{ for } x = \pm 1$$

The critical numbers are -1 and +1

$$\text{Now } f(1) = 1 - 3 + 1 = -1$$



$$f(-1) = -1 + 3 + 1 = 3$$

The stationary points are  $(-1, 3)$  and  $(1, -1)$

iii)  $f(x) = x^{4/5}(x - 4)^2$

we have  $f(x) = x^{4/5}(x - 4)^2$

$$f'(x) = x^{4/5} \cdot 2(x - 4) + \frac{4}{5} x^{-1/5} (x - 4)^2$$

$$= x^{-1/5} (x - 4) \left[ 2x + \frac{4}{5} (x - 4) \right]$$

$$= \frac{2x^{-1/5}}{5} (x - 4) (5x + 2x - 8)$$

$$= \frac{2}{5x^{1/5}} (x - 4) (7x - 8)$$

$f'(x) = 0$  for  $x = 4$  and  $x = \frac{8}{7}$  and  $f'(x)$  does not exist for  $x = 0$ .

$\therefore$  the critical numbers are  $0, 4$  and  $\frac{8}{7}$

$$f(4) = 0, \text{ and } f\left(\frac{8}{7}\right) = \left(\frac{8}{7}\right)^{4/5} \left(\frac{8}{7} - 4\right)^2$$

$$= \left(\frac{8}{7}\right)^{4/5} \left(\frac{-20}{7}\right)^2$$

$\therefore$  the stationary points are  $(4, 0)$  and  $\left(\frac{8}{7}, \left(\frac{8}{7}\right)^{4/5} \left(\frac{-20}{7}\right)^2\right)$

iv)  $f(x) = \frac{x+1}{x^2+x+1}$

we have  $f(x) = \frac{x+1}{x^2+x+1}$

$$\begin{aligned}
 f'(x) &= \frac{(x^2 + x + 1) \cdot 1 - (x + 1)(2x + 1)}{(x^2 + x + 1)^2} \\
 &= \frac{-x^2 - 2x}{(x^2 + x + 1)^2} \\
 &= \frac{-x(x + 2)}{(x^2 + x + 1)^2}
 \end{aligned}$$

Now  $f'(x) = 0$  for  $x = 0$  or  $x = -2$

$\therefore$  the critical numbers are 0 and -2.

$$\text{Also, } f(0) = 1 \text{ and } f(-2) = -\frac{1}{3}$$

Therefore the critical numbers are  $(0, 1)$  and  $(-2, -\frac{1}{3})$

$$\text{v) } f(\theta) = \sin^2 2\theta, 0 \leq \theta \leq \pi$$

$$\begin{aligned}
 f'(\theta) &= 2 \sin 2\theta \cdot 2 \cos 2\theta \\
 &= 2 \sin 4\theta = 0 \text{ for } 4\theta = 0, \pi, 2\pi, 3\pi, 4\pi \\
 &= 0 \text{ for } \theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi
 \end{aligned}$$

$\therefore$  the critical numbers are  $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$$\text{Now } f(0) = 0; f\left(\frac{\pi}{4}\right) = 1; f\left(\frac{\pi}{2}\right) = 0; f\left(\frac{3\pi}{4}\right) = 1 \text{ and } f(\pi) = 0$$

Therefore the stationary points are  $(0, 0), \left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{4}, 1\right)$  and  $(\pi, 0)$

$$\text{vi) } f(\theta) = \theta + \sin \theta, 0 \leq \theta \leq 2\pi$$

$$f'(\theta) = 1 + \cos \theta = 0 \text{ for } \theta = \pi$$

Therefore the critical number is  $\pi$  and

The stationary point is  $(\pi, \pi)$

2. find the absolute maximum and absolute minimum values of  $f$  on the given interval :

Solution:

i)  $(x) = x^2 - 2x + 2, [0, 3]$

$f$  is continuous in  $[0, 3]$

$f'(x) = 2x - 2 = 0$  for  $x = 1$  which belongs to  $[0, 3]$

$\therefore x = 1$  is the only critical point.

the end points are  $x = 0, x = 3$

the values at  $x = 1, x = 0$  and  $x = 3$  are  $f(1), f(0)$  and  $f(3)$  respectively.

i.e., 1, 2, 5

the absolute maximum is 5 and absolute minimum values is 1

ii)  $f(x) = 1 - 2x - x^2; [-4, 1]$

$f$  is continuous in  $[-4, 1]$  ;

$f'(x) = -2 - 2x = 0$  for  $x = -1$  which belongs to  $[-4, 1]$

$\therefore x = -1$  is the only critical point.

The end points are  $x = -4, x = 1$

The values of the function at these points are

$f(-1), f(-4)$  and  $f(1)$  i.e., 2, -7, -2.

The absolute maximum is 2 and absolute minimum values is -7

iii)  $f(x) = x^3 - 12x + 1; [-3, 5]$

$f$  is continuous in  $[-3, 5]$

$$f'(x) = 3x^2 - 12 = 0 \text{ for } x = \pm 2 \text{ which belongs to } [-3, 5]$$

$\therefore$  the critical points are  $x = -2, x = 2$

The end points are  $x = -3, x = 5$

The values of function at these points are  $f(-2), f(2), f(-3)$ , and  $f(5)$

i.e.,  $-8 + 24 + 1, 8 - 24 + 1, -27 + 36 + 1, 125 - 60 + 1$

i.e.,  $17, -15, 10$  and  $66$

The absolute maximum is  $66$  and absolute minimum values is  $-15$

iv)  $f(x) = \sqrt{9 - x^2}; [-1, 2]$

the function is continuous on  $[-1, 2]$

$$f'(x) = \frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{9 - x^2}} = 0 \text{ for } x = 0 \text{ which belongs to } [-1, 2]$$

$\therefore$  the critical points are  $x = 0$

The end points are  $x = -1, x = 2$

The values of function at these points are  $f(0), f(-1)$  and  $f(2)$

i.e.,  $3, 2\sqrt{2}$  and  $\sqrt{5}$

The absolute maximum is  $3$  and absolute minimum values is  $\sqrt{5}$

(v)  $f(x) = \frac{x}{x+1}; [1, 2]$

The function is continuous on  $[1, 2]$

$$f'(x) = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2} \neq 0 \text{ for any point in } [1, 2]$$

$\therefore$  It has no critical point .

The end points are  $x = 1, x = 2$ .

The values of function at these points are  $f(1), f(2)$  i.e.,  $\frac{1}{2}, \frac{2}{3}$

The absolute maximum is  $\frac{2}{3}$  and absolute minimum values is  $\frac{1}{2}$ .

(vi)  $f(x) = \sin x + \cos x ; \left[0, \frac{\pi}{3}\right]$

The function is continuous on  $\left[0, \frac{\pi}{3}\right]$

$$f'(x) = \cos x - \sin x = 0 \text{ for } x = \frac{\pi}{4} \in \left[0, \frac{\pi}{3}\right]$$

$$\therefore \text{the critical points are } x = \frac{\pi}{4}$$

The end points are  $x = 0, x = \frac{\pi}{3}$

The values of function at these points are  $f\left(\frac{\pi}{4}\right), f(0)$  and  $f\left(\frac{\pi}{3}\right)$

i.e.,  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}, 1, \frac{\sqrt{3}}{2} + \frac{1}{2}$  i.e.,  $\sqrt{2}, 1, \frac{\sqrt{3}+1}{2}$

The absolute maximum is  $\sqrt{2}$  and absolute minimum values is 1.

(vii)  $f(x) = x - 2 \cos x ; [-\pi, \pi]$

The function is continuous on  $[-\pi, \pi]$

$$f'(x) = 1 + 2 \sin x = 0 \Rightarrow x = -\frac{\pi}{6}, -\frac{5\pi}{6} \in [-\pi, \pi]$$

$$\therefore \text{the critical points are } -\frac{\pi}{6}, -\frac{5\pi}{6}$$

The end points are  $x = -\pi, x = \pi$

The values of function at these points are  $f\left(-\frac{\pi}{6}\right), f\left(-\frac{5\pi}{6}\right), f(-\pi)$  and  $f(\pi)$

i.e.,  $-\frac{\pi}{6} - \sqrt{3}, -\frac{5\pi}{6} + \sqrt{3}, -\pi + 2, \pi + 2$

The absolute maximum is  $\pi + 2$  and absolute minimum values is  $-\frac{\pi}{6} - \sqrt{3}$ .

3. Find the local maximum and minimum values of the following functions:

Solution:

(i)  $f(x) = x^3 - x$

$$f'(x) = 3x^2 - 1$$

$$f'(x) = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \text{the critical points are } -\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}$$

$$f''(x) = 6x$$

$$\text{At } x = -\frac{1}{\sqrt{3}}, f''(x) < 0$$

$\Rightarrow$  The function attains the local maximum at  $x = -\frac{1}{\sqrt{3}}$

and the local maximum is

$$\begin{aligned} f\left(-\frac{1}{\sqrt{3}}\right) &= -\left(\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}} \end{aligned}$$

$$\text{At } x = \frac{1}{\sqrt{3}}, f''(x) > 0$$

$\Rightarrow$  The function attains the local minimum at  $x = \frac{1}{\sqrt{3}}$

and the local minimum is

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

$$\therefore \text{Local max is } \frac{2}{3\sqrt{3}}$$

$$\text{Local min is } -\frac{2}{3\sqrt{3}}$$

$$(ii) f(x) = 2x^3 + 5x^2 - 4x$$

$$f'(x) = 6x^2 + 10x - 4$$

$$f'(x) = 0 \Rightarrow 6x^2 + 10x - 4 = 0 \Rightarrow x = \frac{1}{3} \text{ or } -2$$

$\therefore$  the critical points are  $-2$ , and  $\frac{1}{3}$

$$f''(x) = 12x + 10$$

$$\text{At } x = -2, f''(x) < 0$$

$\Rightarrow$  The function attains the local maximum at  $x = -2$

and the local maximum is  $f(-2) = 12$

$$\text{At } x = \frac{1}{3}, f''(x) > 0$$

$\Rightarrow$  The function attains the local minimum at  $x = \frac{1}{3}$

and the local minimum is  $f\left(\frac{1}{3}\right) = \frac{2}{27} + \frac{5}{9} - \frac{4}{3} = -\frac{19}{27}$

$\therefore$  Local max is 12

Local min is  $-\frac{19}{27}$

$$(iii) f(x) = x^4 - 6x^2$$

$$f'(x) = 4x^3 - 12x$$

$$f'(x) = 0 \Rightarrow 4x(x^2 - 3) \Rightarrow x = 0, \pm\sqrt{3}$$

$\therefore$  the critical points are  $0, \pm\sqrt{3}$

$$f''(x) = 12x^2 + 12$$

$$\text{At } x = 0, f''(x) < 0$$

$\Rightarrow$  The function attains the local maximum at  $x = 0$

and the local maximum is  $f(0) = 0$

$$\text{At } x = \sqrt{3}, f''(x) > 0$$

$\Rightarrow$  The function attains the local minimum at  $x = \sqrt{3}$

and the local minimum is  $f(\sqrt{3}) = -9$

$$\text{At } x = -\sqrt{3}, f''(x) > 0$$

$\Rightarrow$  The function attains the local minimum at  $x = -\sqrt{3}$

and the local minimum is  $f(-\sqrt{3}) = -9$

iv)  $f(x) = (x^2 - 1)^3$

$$f'(x) = 3(x^2 - 1)^2 \cdot 2x$$

$$f'(x) = 0 \Rightarrow 6(x^2 - 1)^2 \cdot x = 0 \Rightarrow x = 0, \pm 1$$

The critical points are  $0, \pm 1$

$$f''(x) = 6 \{(x^2 - 1)^2 \cdot 1 + x \cdot 2(x^2 - 1)2x\}$$

$$= 6(x^2 - 1)(5x^2 - 1)$$

$$\text{At } x = 0, f''(x) > 0$$

$\Rightarrow$  The function attains the local minimum at  $x = 0$

and the local minimum is  $f(0) = -1$

$$\text{At } x = \pm 1, f''(x) = 0$$

$\Rightarrow$  the second derivative test gives no information about the extreme nature of



at  $x = \pm 1$  and hence extreme values are not known.

v)  $f(\theta) = \sin^2 \theta [0, \pi]$

$$f'(\theta) = 2 \sin \theta \cos \theta = \sin 2\theta$$

$$f'(\theta) = 0 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0, \pi, 2\pi$$

$$\Rightarrow \theta = 0, \frac{\pi}{2}, \pi$$

The critical points are  $0, \frac{\pi}{2}, \pi$

$$f''(\theta) = 2 \cos 2\theta$$

$$\text{At } \theta = \frac{\pi}{2}, f''(\theta) = 2(-1) < 0$$

$\Rightarrow$  The function attains the local maximum at  $\theta = \frac{\pi}{2}$

and the local maximum is  $f\left(\frac{\pi}{2}\right) = 1$

At  $\theta = 0, \pi$  the local min /max do not exist since they are end points of the interval.

vi)  $f(t) = t + \cos t$

$$f'(t) = 1 - \sin t$$

$$f'(t) = 0 \Rightarrow 1 - \sin t = 0$$

$$\Leftrightarrow t = (4n + 1)\frac{\pi}{2}, n \in Z$$

$$\Leftrightarrow f''(t) = -\cos t$$

$$\Leftrightarrow \text{clearly for } t = (4n + 1)\frac{\pi}{2}, f''(t) = 0$$

Since the second derivative vanishes for  $t = (4n + 1)\frac{\pi}{2}$ ,

The extreme nature of  $f$  is not known at the critical numbers.

Exercise 5.10

1. find two numbers whose sum is 100 and whose product is a maximum.

Solution:

Let P denote their product  $P = xy$

To maximize the product, write the product in single variable.

$$P = x(100 - x)$$

$$P' = x(-1) + (100 - x).1 = 100 - 2x$$

$$P'' = -2$$

For max/min,  $P' = 0 \Rightarrow 100 - 2x = 0 \Rightarrow x = 50$

When  $x = 50$ ,  $P'' = -2 < 0$

When  $x = 50$ , the product is maximum.

Thus the numbers are 50, 50

2. Find two positive numbers whose product is 100 and whose sum is minimum.

Solution:

Let  $x$  and  $y$  be the two positive numbers.

$$\therefore xy = 100$$

Let S denote their product

$$S = x + y = x + \frac{100}{x}$$

$$S'(x) = 1 - \frac{100}{x^2} ; \quad S''(x) = \frac{200}{x^3}$$

For max/min,  $S' = 0 \Rightarrow x^2 - 100 = 0 \Rightarrow x = \pm 10$

But  $x$  is positive and hence  $x = 10$

When  $x = 10$ ,  $S'' > 0$

$\therefore S$  is maximum When  $x = 10$

$$y = 10$$

$\therefore$  Thus the numbers are 10, 10

3. Show that of all the rectangles with a given area the one with smallest perimeter is a square.

Solution:

Let  $x$  and  $y$  be the length and breadth of the rectangle.

Let the given area be  $A$ .

$$\therefore A = xy$$

Let  $L$  be the perimeter of the rectangle

$$L = 2x + 2y$$

$$= 2x + 2\left(\frac{A}{x}\right). \text{ note that } A \text{ is constant}$$

$$L''(x) = 2\left[1 - \frac{A}{x^2}\right]$$

$$L''(x) = \frac{4A}{x^3}$$

For max/min,  $L''(x) = 0 \Rightarrow x = \pm\sqrt{A}$

Since  $A$  is the area,  $-\sqrt{A}$  is not possible.  $\therefore x = \sqrt{A}$

When  $x = \sqrt{A}$ , the perimeter is minimum.

Thus  $x = y = \sqrt{A}$

Hence, rectangle of given area with least perimeter must be a square.

4. show that of all the rectangles with a given perimeter the one with the greatest area is a square.

Solution:

Let  $x$  and  $y$  be the length and breadth of the rectangle.

Let  $L$  be the perimeter (given)

$$\therefore L = 2x + 2y$$

Let  $A$  be the area  $A = xy$

$$= x \left[ \frac{L-2x}{2} \right]. \text{ note that } L \text{ is constant}$$

$$A'(x) = \frac{1}{2} [L - 4x]$$

$$A''(x) = -2$$

For max/min  $A'(x) = 0 \Rightarrow L - 4x = 0 \Rightarrow L = 4x$

When  $L = 4x, A''(x) = -2 < 0$

i.e., When  $L = 4x$ , the area is maximum.

i.e.,  $2x + 2y = 4x \Rightarrow x = y$

i.e., the rectangle is a square when area is maximum .

5. find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius  $r$ .

Solution:

Let us take the circle to be a circle with centre  $(0, 0)$  and radius  $r$  and PQRS be the rectangle inscribed in it. Let  $P(x, y)$  be the vertex of the rectangle that lies on the first quadrant. Let  $\theta$  be the angle made by OP with the  $x$ - axis.

$$\text{then } x = r \cos \theta$$

$$y = r \sin \theta$$

Now the dimensions of the rectangle are  $2x = 2r \cos \theta$  ;

$$2y = 2r \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\text{Area of the rectangle } A = 4r^2 \sin \theta \cos \theta = 2r^2 \sin 2\theta$$

$$\text{We want to maximize } A(\theta) = 2r^2 \sin 2\theta$$

$$\text{Now } A'(\theta) = 4r^2 \cos 2\theta = 0 \text{ for } \theta = \frac{\pi}{4}$$

$$A''\left(\frac{\pi}{4}\right) = -8r^2 \times 1 = -8r^2 < 0$$

$$\therefore A \text{ is largest when } \theta = \frac{\pi}{4}$$

$$\text{when } \theta = \frac{\pi}{4}, 2x = 2r \times \frac{1}{\sqrt{2}} = \sqrt{2}r \text{ and}$$

$$2y = 2r \times \frac{1}{\sqrt{2}} = \sqrt{2}r$$

The dimensions of the rectangle are  $\sqrt{2}r$  and  $\sqrt{2}r$

6. Resistance to motion,  $F$ , of a moving vehicle is given by,  $F = \frac{5}{x} + 100x$ .

Determine the minimum value of resistance.

Solution:

$$F = \frac{5}{x} + 100x$$

$$F'(x) = -\frac{5}{x^2} + 100$$

$$F''(x) = \frac{10}{x^3}$$

$$\text{For minimum, } F'(x) = 0 \Rightarrow 100x^2 - 5$$

$$x = \pm \frac{1}{\sqrt{20}}$$

$x = -\frac{1}{\sqrt{20}}$  is not admissible

$$\therefore x = \frac{1}{\sqrt{20}}$$

When  $x = \frac{1}{\sqrt{20}}$

$$F = 5(2\sqrt{5}) + 100 \times \frac{1}{2\sqrt{5}}$$

$$= 10\sqrt{5} + 10\sqrt{5}$$

$$= 20\sqrt{5}$$

### Exercise 5.11

Find the intervals of concavity and the points of inflection of the following functions:

1.  $f(x) = (x - 1)^{\frac{1}{3}}$

we have  $f(x) = (x - 1)^{\frac{1}{3}}$

now  $f'(x) = \frac{1}{3}(x - 1)^{\frac{1}{3}-1} = \frac{1}{3}(x - 1)^{-\frac{2}{3}}$

$$f''(x) = \frac{2}{9}(x - 1)^{-\frac{2}{3}-1} = -\frac{2}{9}(x - 1)^{-\frac{5}{3}} > 0 \text{ for } x < 1 \text{ i.e., } (-\infty, 1)$$

And  $f''(x) < 0$  for  $x > 1$  i.e.,  $(1, \infty)$

Therefore the second derivative test confirms that,  $f(x)$  is concave downward on  $(1, \infty)$  and concave upward on  $(-\infty, 1)$  note that both the first and second order derivatives do not exist at  $x=1$  but  $(1, f(1))$  i.e.,  $(1, 0)$  is a point of inflection.

2.  $f(x) = x^2 - x$

We have  $f(x) = x^2 - x$

$$f'(x) = 2x - 1$$

$$f''(x) = 2 > 0 \text{ for } x \in (-\infty, \infty)$$

$\therefore f(x) = x^2 - x$  is a concave upward on  $(-\infty, \infty)$ .

Since  $f(x)$  does not turn from concave upward to concave downward,

$f(x)$  has no point of inflection.

3.  $f(x) = 2x^3 + 5x^2 - 4x$

We have  $f(x) = 2x^3 + 5x^2 - 4x$

$$f'(x) = 6x^2 + 10x - 4$$

$$f''(x) = 12x + 10$$

$$f''(x) = 0 \Rightarrow x = -\frac{5}{6}$$

$$f''(x) < 0 \text{ for } x \in \left(-\infty, -\frac{5}{6}\right)$$

$$f''(x) > 0 \text{ for } x \in \left(-\frac{5}{6}, \infty\right)$$

Therefore  $f$  is concave upward on  $\left(-\frac{5}{6}, \infty\right)$  and

concave downward on  $\left(-\infty, -\frac{5}{6}\right)$

$$f\left(-\frac{5}{6}\right) = -\frac{125}{108} + \frac{125}{36} + \frac{20}{6} = \frac{610}{108} = \frac{305}{54}$$

Therefore the point of inflection is  $\left(-\frac{5}{6}, f\left(-\frac{5}{6}\right)\right)$  i.e.,  $\left(-\frac{5}{6}, \frac{305}{54}\right)$

4.  $f(x) = x^4 - 6x^2$

We have  $f(x) = x^4 - 6x^2$

$$f'(x) = 4x^3 - 12x$$

$$f''(x) = 12x^2 - 12$$

$$f''(x) = 0 \Rightarrow 12(x^2 - 1) = 0 \Rightarrow x = \pm 1$$

The points divide  $(-\infty, \infty)$  into  $(-\infty, -1)$ ,  $(-1, 1)$  and  $(1, \infty)$

Now  $f''(x) > 0$  for  $x < -1$  and  $x > 1$  i.e., for  $(-\infty, -1)$  and  $(1, \infty)$

i.e.,  $f(x)$  is concave upward on  $(-\infty, -1) \cup (1, \infty)$

and  $f''(x) < 0$  for  $(-1, 1)$

i.e.,  $f(x)$  is concave downward on  $(-1, 1)$  and the points of inflection are  $(-1, f(-1))$ ,  $(1, f(1))$  i.e.,  $(-1, -5)$  and  $(1, -5)$

5.  $f(\theta) = \sin 2\theta$  in  $(0, \pi)$

We have  $f(\theta) = \sin 2\theta$  in  $(0, \pi)$

$$f'(\theta) = 2 \cos 2\theta; \quad f''(\theta) = -4 \sin 2\theta$$

$$f'(\theta) = 0 \Rightarrow 2\theta = 0, \pi, 2\pi$$

$$\text{i.e., } \theta = 0, \frac{\pi}{2}, \pi$$

but  $0, \pi \notin (0, \pi)$

$$\therefore \theta = \frac{\pi}{2} \in (0, \pi)$$

Here  $\frac{\pi}{2}$  divide  $(0, \pi)$  into  $(0, \frac{\pi}{2})$ ,  $(\frac{\pi}{2}, \pi)$ . clearly  $f''(\theta) < 0$  for  $\theta \in (0, \frac{\pi}{2})$  and

$$f''(\theta) > 0 \text{ for } \theta \in (\frac{\pi}{2}, \pi)$$

Hence  $f(\theta)$  is concave downward on  $(0, \frac{\pi}{2})$  and concave upward on  $(\frac{\pi}{2}, \pi)$

And  $(\frac{\pi}{2}, f(\frac{\pi}{2}))$  i.e.,  $(\frac{\pi}{2}, 0)$  is the point of inflection.

6.  $y = 12x^2 - 2x^3 - x^4$

We have  $y = 12x^2 - 2x^3 - x^4$



$$y' = 24x - 6x^2 - 4x^3$$

$$y'' = 24 - 12x - 12x^2$$

$$= -12(x^2 + x - 2)$$

$$= -12(x - 1)(x + 2)$$

$$\text{Now } y'' = 0 \Rightarrow x = -2 \text{ or } x = 1$$

$y'' < 0$  for  $x \in (-\infty, -2)$  and  $x \in (1, \infty)$  And

$$y'' > 0 \text{ for } x \in (-2, 1)$$

Hence  $f$  is concave downward on  $(-\infty, -2) \cup (1, \infty)$  and

$f$  is concave upward on  $(-2, 1)$  and

the points of inflections are  $(-2, f(-2)), (1, f(1))$  i. e.,  $(-2, 48)$

and  $(1, 9)$ .

## 6. DIFFERENTIAL CALCULUS APPLICATIONS-II

### Example sums:

1. If  $y = x^3 + 2x^2$

(i) Find  $dy$

(ii) Find the value of  $dy$  when  $x = 2$  and  $dx = 0.1$

### Solution:

(i) If  $f(x) = x^3 + 2x^2$ ,

then  $f'(x) = 3x^2 + 4x$ , so  $dy = (3x^2 + 4x) dx$

(ii) Substituting  $x = 2$  and  $dx = 0.1$ ,

we get  $dy = (3 \times 2^2 + 4 \times 2)0.1 = 2$ .

2. Compute the values of  $\Delta y$  and  $dy$  if  $y = f(x) = x^3 + x^2 - 2x + 1$

Where  $x$  changes (i) from 2 to 2.05 and (ii) from 2 to 2.01

### Solution:

(i) We have  $f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$

$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$ .

and  $\Delta y = f(2.05) - f(2) = 0.717625$ .

In general  $dy = f'(x) dx = (3x^2 + 2x - 2)dx$

When  $x = 2$ ,  $dx = \Delta x = 0.05$  and

$dy = [(3(2)^2 + 2(2) - 2)] 0.05 = 0.7$

(ii)  $f(2.01) = (2.01)^3 - (2.01)^2 - 2(2.01) + 1 = 9.140701$

$\therefore \Delta y = f(2.01) - f(2) = 0.140701$

When  $dx = \Delta x = 0.01$ ,  $dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$

3. Use differentials to find an approximate value for  $\sqrt[3]{65}$ .

**Solution:**

$$\text{Let } y = f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\text{Then } dy = \frac{1}{3} x^{-\frac{2}{3}} dx$$

Since  $f(64) = 4$ . We get  $x = 64$  and  $dx = \Delta x = 1$

$$\text{This gives } dy = \frac{1}{3} (64)^{-\frac{2}{3}} (1) = \frac{1}{3(16)} = \frac{4}{48}$$

$$\therefore \sqrt[3]{65} = f(64 + 1) = 4 + \frac{1}{48} \approx 4.021$$

4. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of atmost 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere ?

**Solution:**

If the radius of the sphere is  $r$ , then its volume is  $V = \frac{4}{3} \pi r^3$ .

If the error in the measured value of  $r$  is denoted by  $dr = \Delta r$ ,

Then, the corresponding Error in the calculated value of  $V$  is  $\Delta V$ .

which can be approximated by the differential  $dV = 4 \pi r^2 dr$ .

When  $r = 21$  and  $dr=0.05$ ,

This becomes  $dV = 4\pi (21)^2(0.05) \approx 277$ .

The maximum error in the calculated volume is about  $277 \text{ cm}^3 \cdot \sqrt{l}$

5. The time of swing  $T$  of a pendulum is given by  $T = k\sqrt{l}$  where  $k$  is a constant. Determine the percentage error in the time of swing if the length of the pendulum  $l$  changes from 32.1 cm to 32.0 cm.

**Solution:**

$$\text{If } T = k\sqrt{l} = k(l)^{\frac{1}{2}}$$

$$\text{Then } \frac{dT}{dl} = k \left( \frac{1}{2} \times l^{-\frac{1}{2}} \right) = \left( \frac{k}{2\sqrt{l}} \right) \text{ and}$$

$$dl = 32.0 - 32.1 = -0.1 \text{ cm}$$

Error in  $T$  = approximate change in  $T$ .

$$\Delta T \approx dT = \left( \frac{dT}{dl} \right) dl = \left( \frac{k}{2\sqrt{l}} \right) (-0.1)$$

$$\text{Percentage error} = \left( \frac{\Delta T}{T} \right) \times 100\% = \frac{\frac{k}{2\sqrt{l}}(-0.1)}{k\sqrt{l}} \times 100\%$$

$$= \left( \frac{-0.1}{2l} \right) \times 100\% = \left( \frac{-0.1}{2(32.1)} \right) \times 100\%$$

$$= -0.156\%$$

Hence the percentage error in the time of swing is a decrease of 0.156%

6. A circular template has a radius of 10 cm ( $\pm 0.02$ ). Determine the possible error in calculating the area of the templates. Find also the percentage error.

**Solution:**

$$\text{Area of circular template } A = \pi r^2,$$

$$\text{Hence } \frac{dA}{dr} = 2\pi r,$$

$$\text{Approximate change in area } \Delta A \approx (2\pi r)dr.$$

$$\text{When } r = 10 \text{ cm and } dr = 0.02$$

$$\Delta A = (2\pi \cdot 10)(0.02) \approx 0.4\pi \text{ cm}^2$$

i.e, the possible error in calculating the template area is approximately  $1.257 \text{ cm}^2$

$$\begin{aligned} \text{Percentage error} &\approx \left( \frac{0.4\pi}{\pi(10)^2} \times 100 \right) \\ &= 0.4\% \end{aligned}$$

7. Show that the percentage error in the  $n^{\text{th}}$  root of a number is

Approximately  $\frac{1}{n}$  times the percentage error in the number .

**Solution:**

$$\text{Let } x \text{ be the number. Let } y = f(x) = (x)^{\frac{1}{n}}$$

$$\text{Then } \log y = \frac{1}{n} \log x$$

Taking differential on both sides,

$$\text{we have } \frac{1}{y} dy = \frac{1}{n} \times \frac{1}{x} dx$$

$$\text{i.e., } \frac{\Delta y}{y} \approx \frac{1}{y} dy = \frac{1}{n} \cdot \frac{1}{x} dx$$

$$\therefore \frac{\Delta y}{y} \times 100 \approx \frac{1}{n} \left( \frac{dx}{x} \times 100 \right)$$

$=\frac{1}{n}$  times the percentage error in the number.

8. Find the approximate change in the volume  $V$  of a cube of side  $x$  meters caused by increasing the side by 1%

**Solution:**

The volume of the cube of side  $x$  is,

$$V = x^3; dV = 3x^2 dx$$

$$\text{When } dx = 0.01x, dV = 3x^2 \times (0.01x) = 0.03 x^3 m^3.$$

9. Trace the curve  $y = x^3 + 1$

**Solution:**

**(1) Domain, Extent, intercepts and origin:**

The function is defined for all real values of  $x$  and hence the domain is the entire interval  $(-\infty, \infty)$ . Horizontal extent is  $-\infty < x < \infty$  and vertical extent is  $-\infty < y < \infty$ .

Clearly  $x = 0$  yields the  $y$  intercept as  $+1$  and  $y = 0$  yields the  $x$  intercepts as  $-1$ . It is obvious that the curve does not pass through  $(0,0)$ .

**(2) Symmetry Test:** The symmetry test shows that the curve does not possess any of the symmetry properties.

**(3) Asymptotes:** As  $x \rightarrow c$  (for  $c$  finite)  $y$  does not tend to  $\pm \infty$  and vice versa. Therefore the curve does not admit any asymptote.

**(4) Monotonicity:** The first derivative test shows that the curve is increasing throughout  $(-\infty, \infty)$  since  $y' \geq 0$  for all  $x$ .

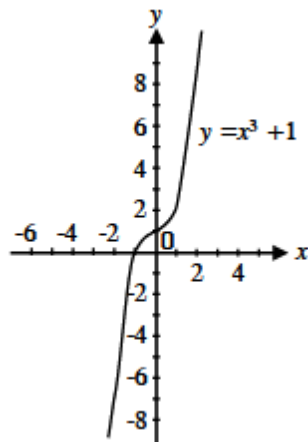
**(5) Special points:** The curve is concave downward in  $(-\infty, 0)$

And concave upward in  $(0, \infty)$

Since  $y'' = 6x < 0$  for  $x < 0$

$y'' = 6x > 0$  for  $x > 0$  and

$y'' = 0$  for  $x = 0$  yields  $(0, 1)$  as the inflection point.



10. Trace the curve  $y^2 = 2x^3$ .

**Solution:**

**(1) Domain, extent, Intercept and Origin :**

When  $x \geq 0$ ,  $y$  is well defined. As  $x \rightarrow \infty$ ,  $y \rightarrow \pm \infty$ ,

The curve exists in first and fourth quadrant only

The intercepts with the axes are given by :

$x = 0$ ,  $y = 0$  and when  $y = 0$ ,  $x = 0$

Clearly the curve passes through origin.

**(2) Symmetry:** By symmetry test, we have, the curve is symmetric about  $x -$  axis only.

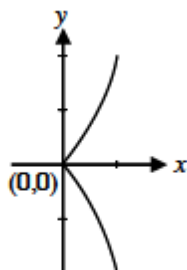
**(3) Asymptotes:** As  $x \rightarrow +\infty$ ,  $y \rightarrow \pm \infty$ , and vice versa.

∴ the curve does not admit asymptotes.

**(4) Monotonicity:** For the branch  $y = \sqrt{2}x^{3/2}$  of the curve is increasing since  $\frac{dy}{dx} > 0$  for  $x > 0$  and the branch  $y = -\sqrt{2}x^{3/2}$  of the curve is decreasing since  $\frac{dy}{dx} < 0$  for  $x > 0$

**(5) Special points:** (0,0) is not a point of inflection.

This curve is called a semi – cubical parabola.



11. Discuss the curve  $y^2 (1 + x) = x^2 (1 - x)$

for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

**Solution:**

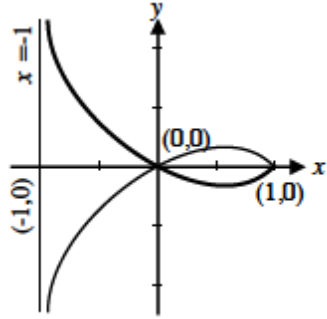
**(i) Existence:** The function is not well defined when  $x > 1$  and  $x \leq -1$  and the curve lies between  $-1 < x \leq 1$ .

**(ii) Symmetry:** The curve is symmetrical about the  $x$  –axis only.

**(iii) Asymptotes:**  $x = -1$  is a vertical asymptote to the curve parallel to  $y$  –axis.

**(iv) Loops:** (0,0) is a point through which the curve passes twice and hence a loop is formed between  $x = 0$  and  $x = 1$ .





12. Discuss the curve  $a^2 y^2 = x^2 (a^2 - x^2)$ ,  $a > 0$

for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

**Solution:**

**(i) Existence:**

The curve is well defined for  $(a^2 - x^2) \geq 0$  i.e.,  $x^2 \leq a^2$

i.e.,  $x \leq a$  and  $x \geq -a$

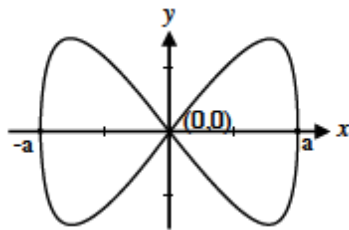
**(ii) Symmetry:** The curve is symmetrical about  $x$ -axis,  $y$ -axis, and hence about the origin.

**(iii) Asymptotes:** It has no asymptote.

**(iv) Loops:** For  $-a < x < 0$  and  $0 < x < a$ ,  $y^2 > 0$

$\Rightarrow y$  is positive and negative

$\therefore$  a loop is formed between  $x = 0$  and  $x = a$  and another loop is formed between  $x = -a$  and  $x = 0$ .



13. Discuss the curve  $y^2 = (x-1)(x-2)^2$ .

for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

**Solution:**

**(i) Existence:**

The curve is not defined for  $x-1 < 0$ ,

ie., whenever  $x < 1$ , the R.H.S. is negative

$\Rightarrow y^2 < 0$  which is impossible.

The curve is defined for  $x \geq 1$ .

**(ii) Symmetry:** The curve is symmetrical about  $x$ -axis.

**(iii) Asymptote:** The curve does not admit asymptotes.

**(iv) Loops:** Clearly a loop is formed between  $(1, 0)$  and  $(2, 0)$ .

14. Determine:  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$  and  $\frac{\partial^2 u}{\partial y \partial x}$

if  $u(x,y) = x^4 + y^3 + 3x^2y^2 + 3x^2y$

**Solution:**

$$\frac{\partial u}{\partial x} = 4x^3 + 6xy^2 + 6xy$$

$$\frac{\partial u}{\partial y} = 3y^2 + 6x^2y + 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 + 6y^2 + 6y$$

$$\frac{\partial^2 u}{\partial y^2} = 6y + 6x^2$$

$$\frac{\partial^2 u}{\partial x \partial y} = 12xy + 6x$$

$$\frac{\partial^2 u}{\partial y \partial x} = 12xy + 6x$$

Note that  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$  due to continuity of  $u$  and its first order partial derivatives.

15. If  $u = \log (\tan x + \tan y + \tan z)$ , prove that  $\Sigma \sin 2x \frac{\partial u}{\partial x} = 2$

**Solution :**

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\begin{aligned} \sin 2x \frac{\partial u}{\partial x} &= \frac{2 \sin x \cos x \cdot \sec^2 x}{\tan x + \tan y + \tan z} \\ &= \frac{2 \tan x}{\tan x + \tan y + \tan z} \end{aligned}$$

Similarly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z}$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{z}{\tan x + \tan y + \tan z}$$

$$\text{L.H.S} = \Sigma \sin 2x \frac{\partial u}{\partial x} = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2 = \text{R.H.S}$$

16. If  $U = (x - y)(y - z)(z - x)$

then show that  $U_x + U_y + U_z = 0$

**Solution:**

$$\begin{aligned} U_x &= (y - z) \{ (x - y)(-1) + (z - x).1 \} \\ &= (y - z) [(z - x) - (x - y)] \end{aligned}$$

Similarly

$$U_y = (z - x) [(x - y) - (y - z)]$$

$$U_z = (x - y) [(y - z) - (z - x)]$$

$$U_x + U_y + U_z = (y - z) [(z - x) - (z - x)] + (x - y) [-(y - z) (y - z)] \\ + (z - x) [(x - y) - (x - y)]$$

$$\therefore U_x + U_y + U_z = 0$$

17. Suppose that  $z = y e^{x^2}$  where  $x = 2t$  and  $v = y \log x$ ,

Find  $\frac{dz}{dt}$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = y e^{x^2} 2x; \quad \frac{\partial z}{\partial y} = e^{x^2}$$

$$\frac{dx}{dt} = 2; \quad \frac{dy}{dt} = -1$$

$$\frac{dz}{dt} = y e^{x^2} 2x (2) + e^{x^2} (-1)$$

$$= e^{x^2} 4xy - e^{x^2}$$

$$= e^{4t^2} [(8t(1 - t) - 1)]$$

$$\therefore \frac{dz}{dt} = e^{4t^2} (8t - 8t^2 - 1)$$

18. If  $w = u^2 e^v$  where  $u = \frac{x}{y}$  and  $v = y \log x$ ,

Find  $\frac{\partial w}{\partial x}$  and  $\frac{\partial w}{\partial y}$

Solution:

We know  $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}$  and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y}$$

$$\frac{\partial w}{\partial u} = 2ue^v; \frac{\partial w}{\partial v} = u^2 e^v;$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}; \quad \frac{\partial u}{\partial y} = \frac{-x}{y^2};$$

$$\frac{\partial v}{\partial x} = \frac{y}{x}; \quad \frac{\partial v}{\partial y} = \log x;$$

$$\therefore \frac{\partial w}{\partial x} = \frac{2ue^v}{y} + u^2 e^v \frac{y}{x}$$

$$= x^y \frac{x}{y^2} (2 + y)$$

$$\therefore \frac{\partial w}{\partial y} = 2ue^v \frac{-x}{y^2} + u^2 e^v \log x$$

$$= \frac{x^2}{y^3} x^y [y \log x - 2]$$

19.  $w = x + 2y + z^2$  and  $x = \cos t$ ;  $y = \sin t$ ;  $z = t$ .

Find  $\frac{dw}{dt}$

Solution:

We know  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$

$$\frac{\partial w}{\partial x} = 1; \quad \frac{dx}{dt} = -\sin t$$

$$\frac{\partial w}{\partial y} = 2; \quad \frac{dy}{dt} = \cos t$$

$$\frac{\partial w}{\partial z} = 2z; \frac{dz}{dt} = 1$$

$$\begin{aligned}\therefore \frac{dw}{dt} &= 1(-\sin t) + 2 \cos t + 2z \\ &= -\sin t + 2 \cos t + 2t\end{aligned}$$

20. Verify Euler's theorem for  $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$

**Solution:**

$$f(tx, ty) = \frac{1}{\sqrt{t^2x^2+t^2y^2}} = \frac{1}{t} f(x, y) = t^{-1} f(x, y)$$

$\therefore f$  is a homogenous function of degree  $-1$  and

By Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -f$$

**Verification:**

$$f_x = -\frac{1}{2} \frac{2x}{(x^2+y^2)^{3/2}} = \frac{-x}{(x^2+y^2)^{3/2}}$$

Similarly,

$$f_y = \frac{-y}{(x^2+y^2)^{3/2}}$$

$$xf_x + yf_y = -\frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{-1}{\sqrt{x^2+y^2}} = -f.$$

Hence Euler's theorem is verified.

21. If  $u$  is a homogenous function of  $x$  and  $y$  of degree  $n$ ,

prove that  $x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y}$

**Solution:**

Since  $U$  is a homogeneous function in  $x$  and  $y$  of degree  $n$ ,

$U_y$  is homogeneous function in  $x$  and  $y$  of degree  $n - 1$ .

Applying Euler's theorem for  $U_y$

we have,

$$x (U_y)_x + y (U_y)_y = (n - 1) U_y$$

$$\text{i.e., } xU_{yx} + yU_{yy} = (n - 1) U_y$$

$$x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n - 1) \frac{\partial u}{\partial y}$$

22. Using Euler's theorem, prove that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$

$$\text{If } u = \sin^{-1} \left( \frac{x-y}{\sqrt{x} + \sqrt{y}} \right)$$

Solution:

R.H.S. is homogeneous and hence define

$$f = \sin u = \left( \frac{x - y}{\sqrt{x} + \sqrt{y}} \right)$$

$\Rightarrow f$  is homogenous of degree  $\frac{1}{2}$

By Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{2} f$$

$$x \cdot \frac{\partial}{\partial x} (\sin u) + y \cdot \frac{\partial}{\partial y} (\sin u) = \frac{1}{2} \sin u$$

$$x \cdot \frac{\partial u}{\partial x} (\cos u) + y \cdot \frac{\partial u}{\partial y} (\cos u) = \frac{1}{2} \sin u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$$

## 6. DIFFERENTIAL CALCULUS

### APPLICATION-II

#### Exercise sums

#### EXERCISE 6.1

1. Find the differential of the functions

$$(i) \quad y = x^5 \quad (ii) \quad y = \sqrt[4]{x} \quad (iii) \quad y = \sqrt{x^4 + x^2 + 1}$$
$$(iv) \quad y = \frac{x-2}{2x+3} \quad (v) \quad y = \sin 2x \quad (vi) \quad y = x \tan x$$

Solution :

$$(i) \quad y = f(x) = x^5$$

$$f'(x) = 5x^4 \Rightarrow dy = 5x^4 dx$$

$$(ii) \quad y = f(x) = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4} x^{-\frac{3}{4}} \Rightarrow dy = \frac{1}{4} \cdot x^{-\frac{3}{4}} dx$$

$$(iii) \quad y = f(x) = (x^4 + x^2 + 1)^{1/2}$$

$$f(x) = \frac{1}{2} (x^4 + x^2 + 1)^{-\frac{1}{2}} (4x^3 + 2x)$$

$$dy = \frac{4x^3 + 2x}{2(x^4 + x^2 + 1)^{\frac{1}{2}}} dx = \frac{x(2x^2 + 1)}{\sqrt{x^4 + x^2 + 1}} dx$$

$$(iv) \quad y = f(x) = \frac{x-2}{2x+3}$$

$$f'(x) = \frac{(2x+3) \cdot 1 - (x-2) \cdot 2}{(2x+3)^2}$$



$$= \frac{7}{(2x + 3)^2}$$

$$dy = \frac{7}{(2x + 3)^2} dx$$

(v)  $y = f(x) = \sin 2x$

$$f'(x) = 2 \cos 2x$$

$$\implies dy = 2 \cos 2x \cdot dx$$

(vi)  $y = f(x) = x \tan x$

$$f'(x) = x \cdot \sec^2 x + \tan x \cdot 1$$

$$\implies dy = (x \sec^2 x + \tan x) dx$$

2. Find the differential  $dy$  and evaluate  $dy$  for the given values of  $x$  and  $dx$ .

(i)  $y = 1 - x^2, \quad x = 5, \quad dx = \frac{1}{2}$

(ii)  $y = x^4 - 3x^3 + x - 1, \quad x = 2, \quad dx = 0.1.$

(iii)  $y = (x^2 + 5)^3, \quad x = 1, \quad dx = 0.05$

(iv)  $y = \sqrt{1 - x}, \quad x = 0, \quad dx = 0.02$

(v)  $y = \cos x, \quad x = \frac{\pi}{6}, \quad dx = 0.05$

**Solution :**

(i)  $dy = f'(x)dx$

$$y = f(x) = (1 - x^2)$$
$$dy = -2x dx$$

When  $x = 5$ ,  $dx = 1/2$ ;

$$dy = -2 \times 5 \times \frac{1}{2}$$
$$= -5$$

(ii)  $y = f(x) = x^4 - 3x^3 + x - 1$

$$dy = (4x^3 - 9x^2 + 1) dx$$

When  $x = 2$ ,  $dx = 0.1$ ;

$$dy = (4 \times 8 - 9 \times 4 + 1) (0.1)$$
$$= -0.3$$

(iii)  $y = f(x) = (x^2 + 5)^3$

$$dy = 3(x^2 + 5)^2 2x dx$$
$$= 6(x^2 + 5)^2 x dx$$

When  $x = 1$ ,  $dx = 0.05$ ,

$$dy = 6(1+5)^2 (1) (0.05)$$
$$= 10.8$$

(iv)  $y = f(x) = (1-x)^{1/2}$

$$dy = \frac{1}{2} (1 - x)^{-\frac{1}{2}} (-1) dx$$

$$= -\frac{1}{2\sqrt{1-x}}dx$$

When  $x = 0$ ,  $dx = 0.02$ ;

$$\begin{aligned} dy &= -\frac{1}{2}(1)^{-\frac{1}{2}}(0.02) \\ &= -0.01 \end{aligned}$$

(v)  $y = f(x) = \cos x$   
 $dy = -\sin x dx$

When  $x = \frac{\pi}{6}$ ,  $dx = 0.05$ ;

$$dy = -\sin \frac{\pi}{6} \times 0.05 = -0.025$$

3. Use differentials to find an approximate value for the given number

(i)  $\sqrt{36.1}$     (ii)  $\frac{1}{10.1}$     (iii)  $y = \sqrt[3]{1.02} + \sqrt[4]{1.02}$     (iv)  $(1.97)^6$

Solutions:

(i) Let  $y = f(x) = x^{\frac{1}{2}}$

Take  $x = 36$ ,  $dx = \Delta x = 0.1$

$$\begin{aligned} dy &= \frac{1}{2}(x)^{-\frac{1}{2}} dx \\ &= \frac{1}{2}(36)^{-\frac{1}{2}} \times 0.1 \\ &= \frac{0.1}{12} = 0.0083 \end{aligned}$$

$$f(x + \Delta x) = y + dy = f(36) + 0.008 = 6 + 0.008$$

$$\sqrt{36.1} \cong 6.008$$

(ii) Let  $y = f(x) = \frac{1}{x}$

Take  $x = 10, dx = \Delta x = 0.1$

$$\begin{aligned} dy &= -\frac{1}{x^2} dx \\ &= -\frac{1}{100} \times 0.1 \\ &= -0.001 \end{aligned}$$

$$f(x + \Delta x) = y + dy = f(10) - 0.001$$

$$= \frac{1}{10} - 0.001$$

$$= 0.099$$

$$\frac{1}{10.1} \cong 0.099$$

(iii) Let  $y = f(x) = x^{\frac{1}{3}}$

Take  $x = 1; dx = \Delta x = 0.02$

$$\begin{aligned} dy &= \frac{1}{3} x^{-\frac{2}{3}} \cdot dx \\ &= \frac{1}{3} \times (1) (0.02) \\ &= 0.0066 \end{aligned}$$

$$f(x+\Delta x) = y + dy$$

$$= f(1) + 0.0066$$

$$= 1 + 0.0066$$

$$(1.02)^{\frac{1}{3}} = 1.0066$$

$$\text{Again, let } y = f(x) = x^{\frac{1}{3}}$$

$$\text{Here } x = 1, dx = \Delta x = 0.02$$

$$dy = \frac{1}{3}x^{-\frac{2}{3}} dx$$

$$= \frac{1}{3} (1) (0.02)$$

$$= 0.0066$$

$$f(x+\Delta x) = y + dy = f(1) + 0.0066 = 1.0066$$

$$(1.02)^{\frac{1}{4}} = 1.005$$

$$(1.02)^{\frac{1}{3}} + (1.02)^{\frac{1}{4}} = 1.0066 + 1.005$$

$$= 2.0116$$

(iv) Let  $y = f(x) = x^6$

$$\text{Take } x = 2, dx = \Delta x = -0.03$$

$$dy = 6x^5 dx$$

$$= 6 \times 2^5 \times (-0.03)$$

$$= -5.76$$

$$f(x+\Delta x) = y + dy = f(2) - 5.76$$

$$= 2^6 - 5.76 = 58.24$$

$$(1.97)^6 = 58.24$$

4. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing (i) the volume of the cube and (ii) the surface area of cube.

**Solution:**

(i) Volume of the cube  $v = a^3 \Rightarrow \frac{dv}{da} = 3a^2$   
 Approximate change in the volume  $\Delta v \approx 3a^2 da$   
 $\Delta v = 3 \times 30 \times 30 \times 0.1 = 270 \text{ cm}^3$   
 The maximum possible error in volume is  $270 \text{ cm}^3$

(ii) Surface area of the cube  $S = 6a^2$   
 $\frac{dS}{da} = 12a$   
 Approximate change in the surface area  
 $\Delta S = 12a da$   
 $= 12 \times 30 \times 0.1$   
 $= 36 \text{ cm}^2$

The maximum possible error in the surface area =  $36 \text{ cm}^2$

5. The radius of a circular disc is given as 24 cm with a maximum error in measurement of 0.02 cm.

- (i) Use differentials to estimate the maximum error in the calculated area of the disc
- (ii) Compute the relative error?

**Solution:**

(i) Area of the disc  $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

Approximate change in the area  $\Delta A \approx 2\pi r dr$

$$= 2\pi(24)(0.02) = 0.96\pi \text{ cm}^2$$

The maximum error in the area =  $0.96\pi \text{ cm}^2$

(ii)  $A = \pi r^2$

Taking log on both sides,

$$\log A = \log \pi + 2 \log r$$

Taking differential on both sides,

$$\frac{1}{A} dA = 2 \cdot \frac{1}{r} dr$$

The relative error in A

$$\text{i.e., } \frac{\Delta A}{A} = \frac{1}{A} dA = 2 \cdot \frac{1}{r} dr$$

$$= 2 \cdot \frac{1}{24} \times (0.02)$$

$$= 0.0017$$

The relative error in A is approximately 0.0017

### **EXERCISE 6.2**

1. Trace the curve  $y = x^3$

**Solution :**

(i) **Domain, extent, intercepts and origin**

The function is defined for all real values of  $x$  and hence the domain is  $(-\infty, \infty)$ . The horizontal extent is  $-\infty < x < \infty$  and the vertical extent is  $-\infty < y < \infty$ . Clearly it passes through the origin since  $(0,0)$  satisfies the equation.

**(ii) Symmetry**

It is symmetrical about the origin.

**(iii) Asymptotes**

The curve does not admit any asymptote.

**(iv) Monotonicity**

Since  $y \geq 0$  for all  $x$ , the curve is increasing in  $(-\infty, \infty)$ .

**(v) Special points**

Since  $y'' = 6x$ , the curve is concave upward in  $(0, \infty)$  and convex upward in  $(-\infty, 0)$

$y'' = 0$  for  $x = 0$  yields  $(0,0)$  as the point of inflection.

**Discuss the following curves for**

(i) Existence (ii) Symmetry (iii) Asymptotes (iv) loops

$$2.y^2 = x^2 (1-x^2)$$

**Solution :**

**(i) Existence**

The curve is defined only for  $(1 - x^2) \geq 0$  i.e.,  $x \leq 1$  and



$$x \geq -1$$

**(ii) Symmetry**

The curve is symmetrical about  $x$ -axis and  $y$ -axis and hence about the origin.

**(iii) Asymptotes**

It has no asymptotes

**(iv) Loops :**

(0,0) is a point through which the curve passes twice.

For  $-1 < x < 0$  and  $0 < x < 1$ ,  $y^2 > 0$  i.e.,  $y$  is positive and negative.

Two loops are formed between  $x = -1$  and  $x = 0$ ;  $x = 0$  and  $x = 1$ .

3.  $y^2 (2 + x) = x^2 (6-x)$

**Solution:**

**(i) Existence**

The function is well defined only for  $x \leq 6$   
and  $x > -2$  i.e., the curve lies between  $-2 < x \leq 6$

**(ii) Symmetry**

The curve is symmetrical about  $x$  - axis.

**(iii) Asymptotes**

When  $x = -2$ ,  $y^2$  becomes infinite i.e.,  $y \rightarrow \pm\infty$   
 $x = -2$  is a vertical asymptote.

(iv) **Loops:**  $(0, 0)$  is a point through which the curve passes twice

and hence a loop is formed between  $x = 0$  and  $x = 6$ .

$$(4) \quad y^2 = x^2 (1-x)$$

**Solution:**

(i) **Existence**

The curve is defined only for  $(1-x) \leq 0$  i.e.,  $x \leq 1$

(ii) **Symmetry**

The curve is symmetrical about  $x$  – axis.

(iii) **Asymptotes**

It has no asymptotes.

(iv) **Loops**

$(0, 0)$  is a point through which the curve passes twice and hence a loop is formed between  $x = 0$  and  $x = 1$ .

$$5.y^2 = (x - a) (x - b)^2 \quad ; \quad a, b > 0, \quad a > b,$$

**Solution:**

(i) **Existence**

The curve is defined only for  $x = b$  and  $x \geq a$ . for  $x = b$ , we have a point only.

(ii) **Symmetry**

The curve is symmetrical about  $x$  – axis.

(iii) **Asymptotes**

It has no asymptotes.

(iv) **Loops**

There is no loop

EXERCISE 6.3

1. verify  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$  for the following functions:

(i)  $u = x^2 + 3xy + y^2$  (ii)  $u = \frac{x}{y^2} - \frac{y}{x^2}$  (iii)  $u = \sin 3x \cos 4y$

(iv)  $u = \tan^{-1} \left( \frac{x}{y} \right)$

Solution:

(i)  $u = x^2 + 3xy + y^2$

$$\frac{\partial u}{\partial x} = 2x + 3y; \quad \frac{\partial u}{\partial y} = 3x + 2y$$

$$\frac{\partial^2 u}{\partial x \partial y} = 3; \quad \frac{\partial^2 u}{\partial y \partial x} = 3$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(ii)  $u = \frac{x}{y^2} - \frac{y}{x^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{y^2} - y(-2).x^{-3} = \frac{1}{y^2} + \frac{2y}{x^3} = \frac{x^3 + 2y^3}{x^3 y^2}$$

$$\frac{\partial u}{\partial y} = x.(-2).y^{-3} - \frac{1}{x^2} = -\frac{2x}{y^3} - \frac{1}{x^2} = -\left( \frac{y^3 + 2x^3}{x^2 y^3} \right)$$

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = -\frac{2}{y^3} + \frac{2}{x^3}; \quad \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = -\frac{2}{y^3} + \frac{2}{x^3} \cdot 1$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$(iii) u = \sin 3x \cos 4y$$

$$\frac{\partial u}{\partial x} = \cos 4y \cdot \cos 3x \cdot 3 = 3 \cos 3x \cos 4y$$

$$\frac{\partial u}{\partial y} = \sin 3x (-\sin 4y) \cdot 4 = -4 \sin 3x \sin 4y$$

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = -4 \sin 4y [\cos 3x \cdot 3] = -12 \cos 3x \sin 4y$$

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = 3 \cos 3x [-\sin 4y \cdot 4] = -12 \cos 3x \sin 4y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$(iv) u = \tan^{-1} \left( \frac{x}{y} \right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left( \frac{x}{y} \right)^2} \cdot \left( \frac{1}{y} \right) = \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1 + \left( \frac{x}{y} \right)^2} \cdot x \left( -\frac{1}{y^2} \right) = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = - \left[ \frac{(x^2 + y^2) \cdot 1 - (x) \cdot 2x}{(x^2 + y^2)^2} \right] = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = - \left[ \frac{(x^2 + y^2) \cdot 1 - (y) \cdot 2y}{(x^2 + y^2)^2} \right] = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

2. (i) if  $u = \sqrt{x^2 + y^2}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$

(ii) if  $u = e^{\frac{x}{y}} \sin \frac{x}{y} + e^{\frac{y}{x}} \cos \frac{y}{x}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Solution :

$$u = \sqrt{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} (2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{x^2}{\sqrt{x^2 + y^2}} + \frac{y^2}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2} = u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$$

(ii) the given function is homogeneous and therefore it is easy to prove by Euler's theorem.

$$u = e^{\frac{x}{y}} \sin \frac{x}{y} + e^{\frac{y}{x}} \cos \frac{y}{x}$$

$$u(tx, ty) = e^{\frac{tx}{ty}} \sin \frac{tx}{ty} + e^{\frac{ty}{tx}} \cos \frac{ty}{tx} = u(x, y)$$

⇒ degree of  $u$  is 0

By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu = 0. u = 0$$

3. Using chain rule find  $\frac{dw}{dt}$  for each of the following:

(i)  $w = e^{xy}$  where  $x=t^2$ ,  $y=t^3$

(ii)  $w = \log(x^2+y^2)$  where  $x=e^t$ ,  $y=e^{-t}$

(iii)  $w = \frac{x}{(x^2+y^2)}$  where  $x=\cos t$ ,  $y=\sin t$ .

(iv)  $w = xy + z$  where  $x=\cos t$ ,  $y=\sin t$ ,  $z=t$

**Solution:**

$$W=e^{xy}$$

$$(i) \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= e^{xy} \cdot y \cdot 2t + e^{xy} \cdot x \cdot 3t^2$$

$$= e^{xy} [t^3 \cdot 2t + t^2 \cdot 3t^2] = w[5t^4] = 5e^{t^5} t^4$$

$$(ii) \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= \frac{1}{x^2+y^2} \cdot 2xe^t + \frac{1}{x^2+y^2} \cdot 2y(-e^{-t})$$

$$= \frac{2}{e^{2t}+e^{-2t}} [e^{2t} - e^{-2t}]$$

$$(iii) \frac{dw}{dt} = \frac{(x^2+y^2) \cdot 1 - x \cdot 2x}{(x^2+y^2)^2} \cdot (-\sin t) + \frac{0 - x \cdot 2y}{(x^2+y^2)^2} (\cos t)$$

$$= \frac{1}{(x^2+y^2)^2} [(x^2 - y^2)(-\sin t) - 2xy \cos t]$$

$$= \frac{1}{1} [(\sin^2 t - \cos^2 t)(-\sin t) - 2(\cos t)(\sin t) \cos t]$$

$$= \sin t [-\sin^2 t + \cos^2 t - 2 \cos^2 t]$$

$$= -\sin t$$

$$(iv) \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= y.(-\sin t) + x.(\cos t) + 1.1$$

$$= -\sin^2 t + \cos^2 t + 1 = 2 \cos^{-1} t$$

4. (i) find  $\frac{\partial w}{\partial r}$  and  $\frac{\partial w}{\partial \theta}$  if  $w = \log(x^2 + y^2)$  where  $x = r \cos \theta$ ,  $y = r \sin \theta$

(ii) find  $\frac{\partial w}{\partial u}$  and  $\frac{\partial w}{\partial v}$  if  $w = x^2 + y^2$

$$\text{where } x = u^2 - v^2, y = 2uv$$

(iii) find  $\frac{\partial w}{\partial u}$  and  $\frac{\partial w}{\partial v}$  if  $w = \sin^{-1} xy$

Where  $x = u + v, y = u - v$ .

Solution:

$$(i) \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{1}{x^2 + y^2} \cdot 2x(\cos \theta) + \frac{1}{x^2 + y^2} \cdot 2y \cdot \sin \theta$$

$$= \frac{2}{r^2} [r \cos^2 \theta + r \sin^2 \theta] = \frac{2}{r}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = \frac{1}{x^2 + y^2} \cdot 2x(-r \sin \theta) + \frac{1}{x^2 + y^2} \cdot 2y \cdot (r \cos \theta)$$

$$= \frac{2}{r^2} [-x \sin \theta + y \sin \theta]$$

$$= \frac{2}{r} [-r \sin \theta \cos \theta + r \sin \theta \cos \theta] = 0$$

(ii)  $w = x^2 + y^2$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$$

$$= 2x \cdot 2u + 2y \cdot 2v$$

$$= 4[(u^2 - v^2)u + 2uv \cdot v]$$

$$= 4[u^3 + uv^2] = 4u[u^2 + v^2]$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= 2x \cdot (-2v) + 2y \cdot (2u)$$

$$= -4[(u^2 - v^2)v - 2uv \cdot u]$$

$$= -4[u^2v - v^3 - 2u^2v]$$

$$= 4[v^3 + u^2v] = 4v[u^2 + v^2]$$

(iii)  $w = \sin^{-1} xy$  Where  $x = u + v, y = u - v$ .

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$$

$$= \frac{1}{\sqrt{1-(xy)^2}} \cdot y \cdot 1 + \frac{1}{\sqrt{1-(xy)^2}} \cdot x \cdot 1$$

$$= \frac{x+y}{\sqrt{1-(xy)^2}} = \frac{2u}{\sqrt{1-(u^2-v^2)^2}}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$



$$= \frac{1}{\sqrt{1-(xy)^2}} \cdot y \cdot 1 + \frac{1}{\sqrt{1-(xy)^2}} \cdot x \cdot (-1)$$

$$= \frac{y-x}{\sqrt{1-(xy)^2}} = \frac{-2v}{\sqrt{1-(u^2-v^2)^2}}$$

5. using Euler's theorem prove the following:

(i) if  $u = \tan^{-1} \left( \frac{x^3+y^3}{x-y} \right)$ ,

Show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$ .

(ii)  $u = xy^2 \sin \left( \frac{x}{y} \right)$ ,

Show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u$ .

(iii) if  $u$  is a homogeneous function of  $x$  and  $y$  of degree  $n$ ,

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y \partial x}$$

prove that  $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1) \frac{\partial u}{\partial x}$

(iv) if  $V = ze^{ax+by}$  and  $z$  is a homogeneous function of degree  $n$

in  $x$  and  $y$  prove that  $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = (ax + by + n)V$ .

**Solution:** (i)

$u$  is not a homogenous function. but  $\tan u$  is a homogeneous function.

$$\therefore \text{define } f = \tan u = \frac{x^3+y^3}{x-y}$$

$\Rightarrow f$  is a homogeneous function of degree 2.

⇒ By euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u.$$

(ii)  $u = xy^2 \sin\left(\frac{x}{y}\right),$

$$u(tx, ty) = t^3 xy^2 \sin\left(\frac{x}{y}\right)$$

⇒  $u$  is a homogeneous function of degree 3.

⇒ By euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u$$

(iii) since  $U$  is a homogeneous function of degree  $n$ ,

$U_x$  is a homogeneous function of degree  $n - 1$ .

$$x \cdot (U_x)_x + y (U_x)_y = (n - 1)(U_x)$$

$$x \cdot (U_{xx}) + y (U_{xy}) = (n - 1)U_x$$

$$\text{i.e., } x \cdot \frac{\partial^2 U}{\partial x^2} + y \frac{\partial^2 U}{\partial x \partial y} = (n - 1) \frac{\partial U}{\partial x}$$

(iv)  $V = ze^{ax+by}$

$$x \frac{\partial V}{\partial x} = x \left[ z \cdot e^{ax+by} \cdot a + e^{ax+by} \cdot \frac{\partial Z}{\partial x} \right]$$

$$y \frac{\partial V}{\partial y} = y \left[ z \cdot e^{ax+by} \cdot b + e^{ax+by} \cdot \frac{\partial Z}{\partial y} \right]$$

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = e^{ax+by} \left[ axz + byz + x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} \right]$$
$$= e^{ax+by} [axz + byz + nz]$$

$$\text{since } x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = nz$$

$$\therefore x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = ze^{ax+by} (ax + by + n)$$

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = (ax + by + n)V$$

**END**